Mathematical Induction

Part Two

Outline for Today

- ***Variations on Induction***
  - Starting later, taking different step sizes, and more!
- ***Complete Induction***
  - When one assumption isn't enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

***Proof*:** Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

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New Stuff!

Variations on Induction:

Starting Later

Step Sizes Other than +1

Induction Starting at 0

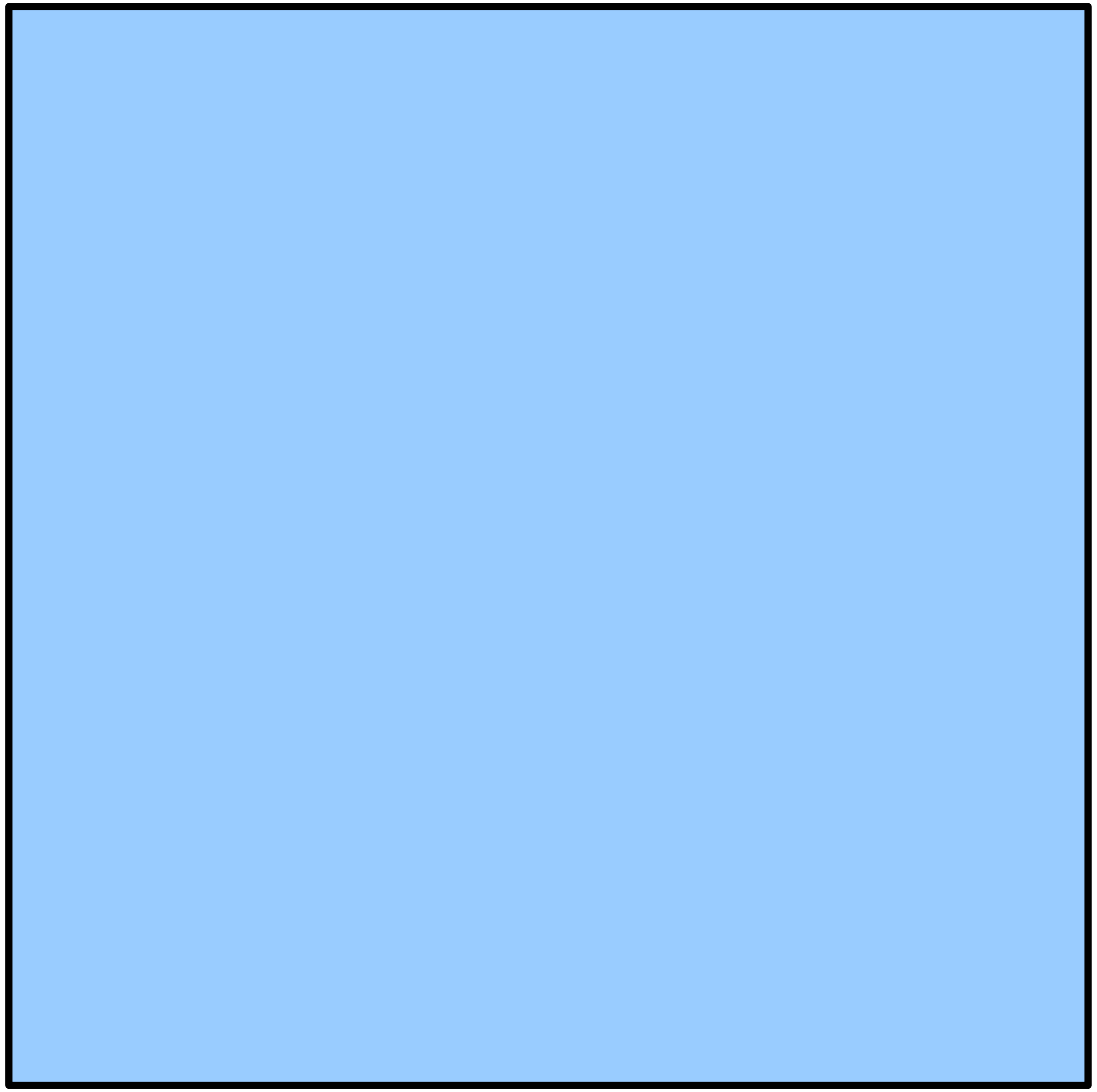
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for an arbitrary $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for an arbitrary $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



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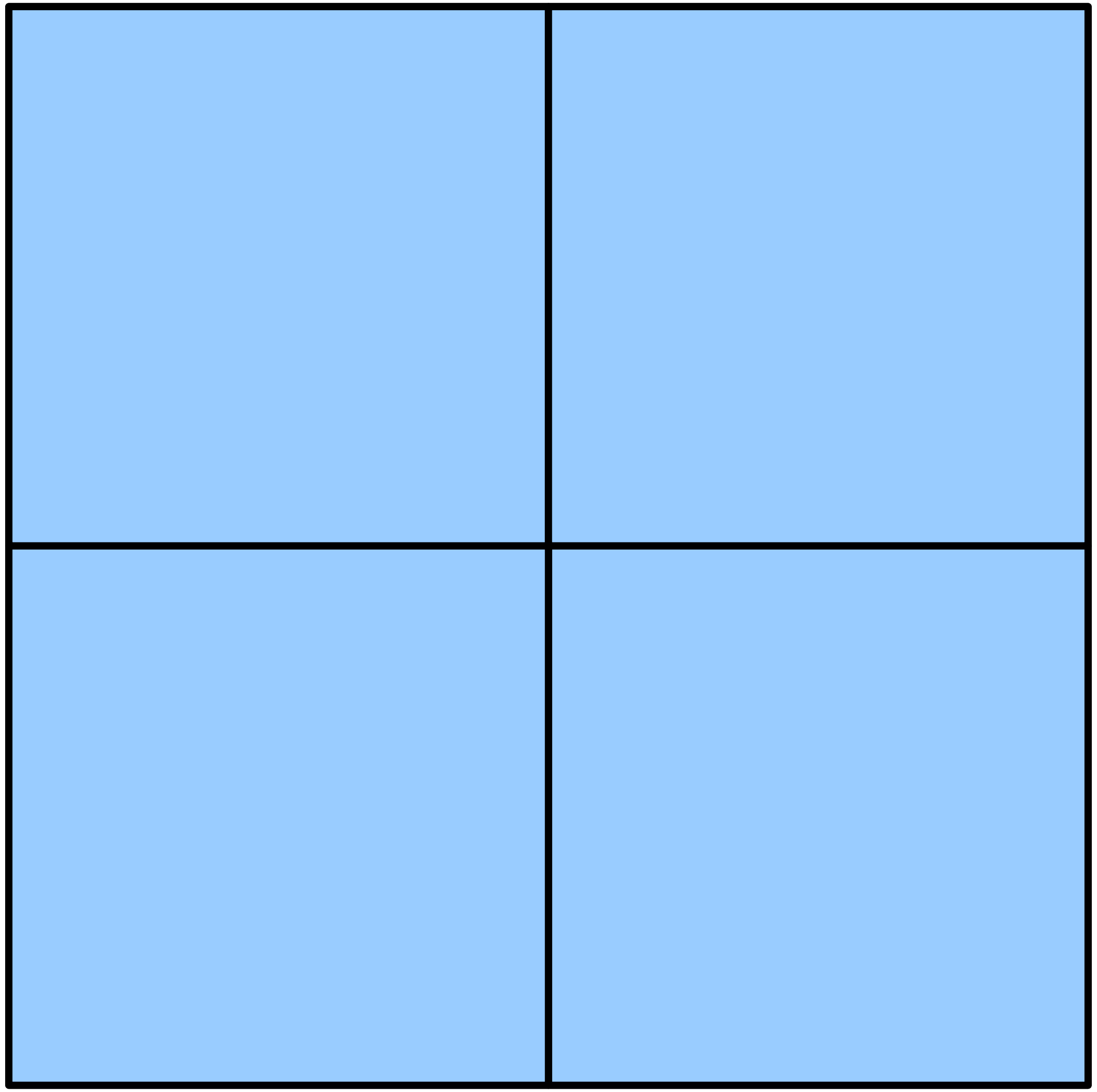

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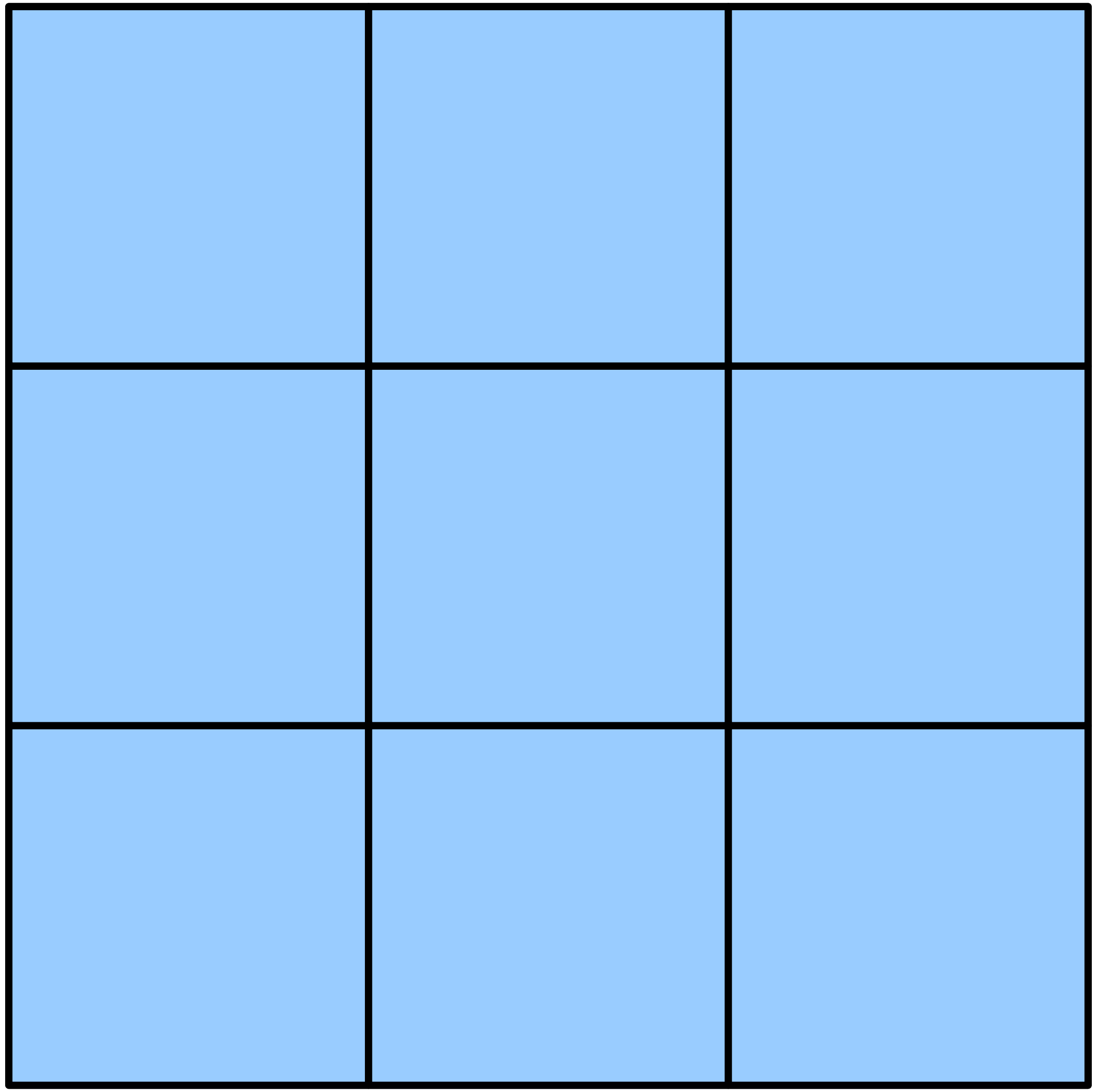

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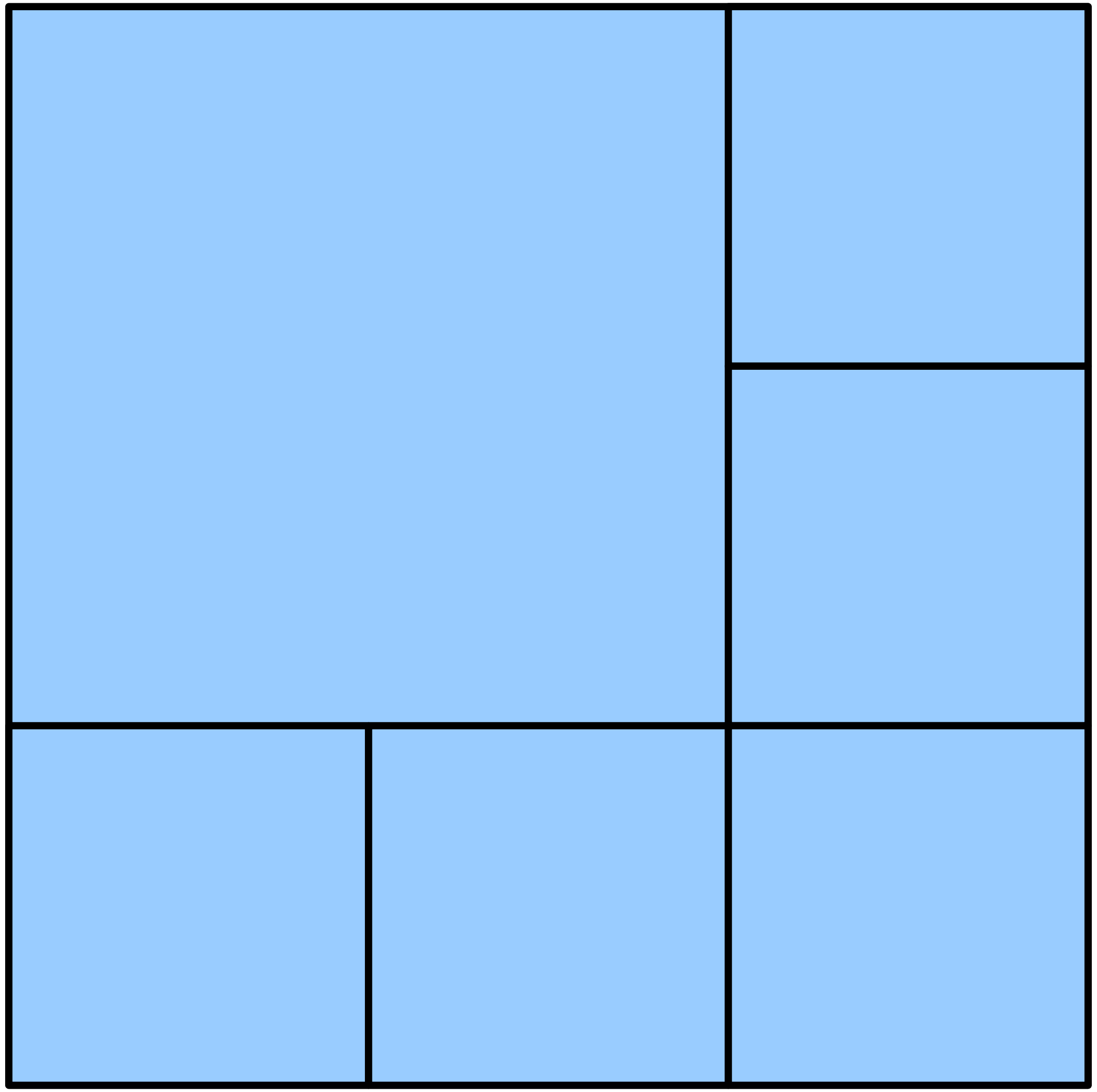

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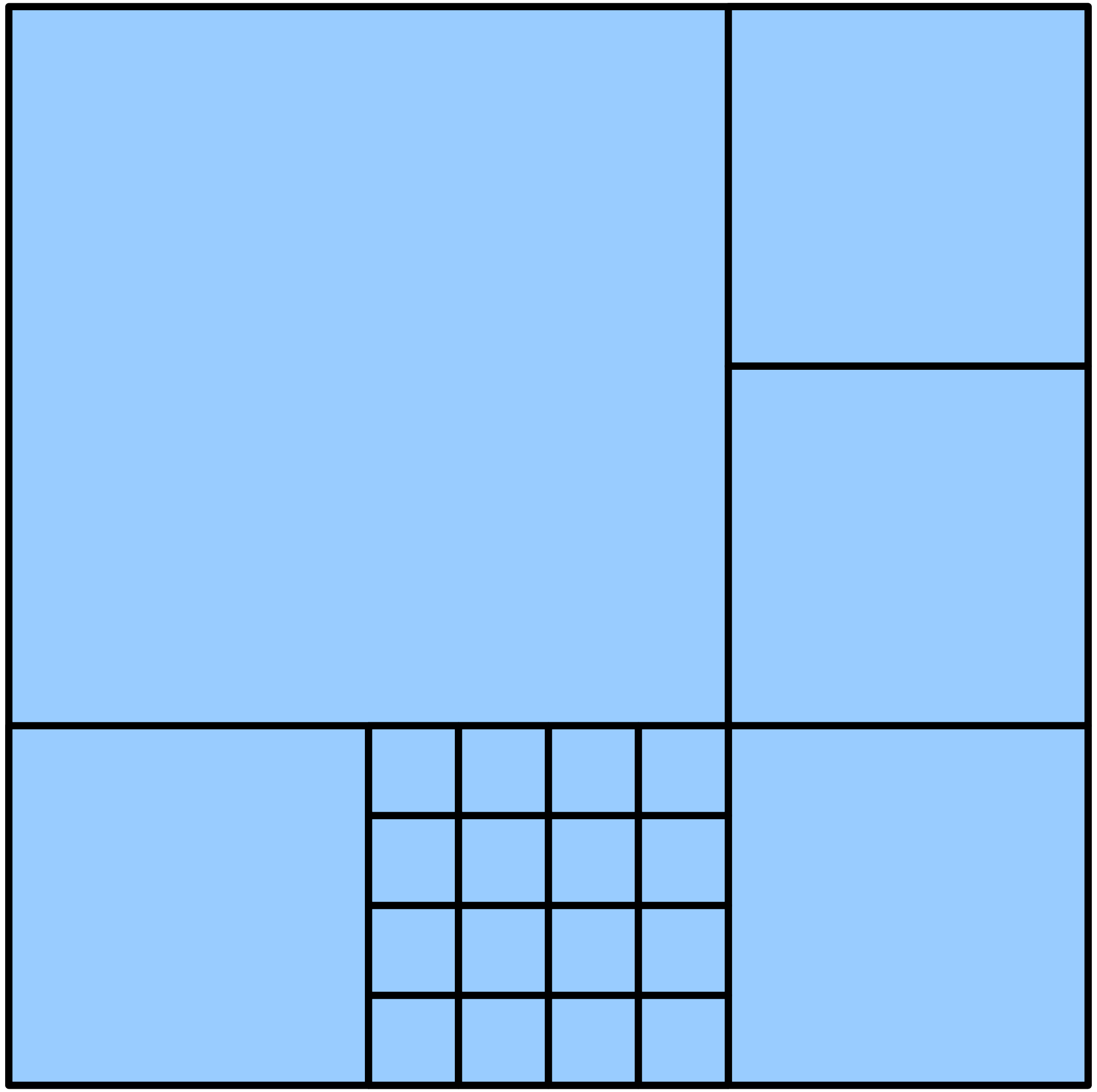

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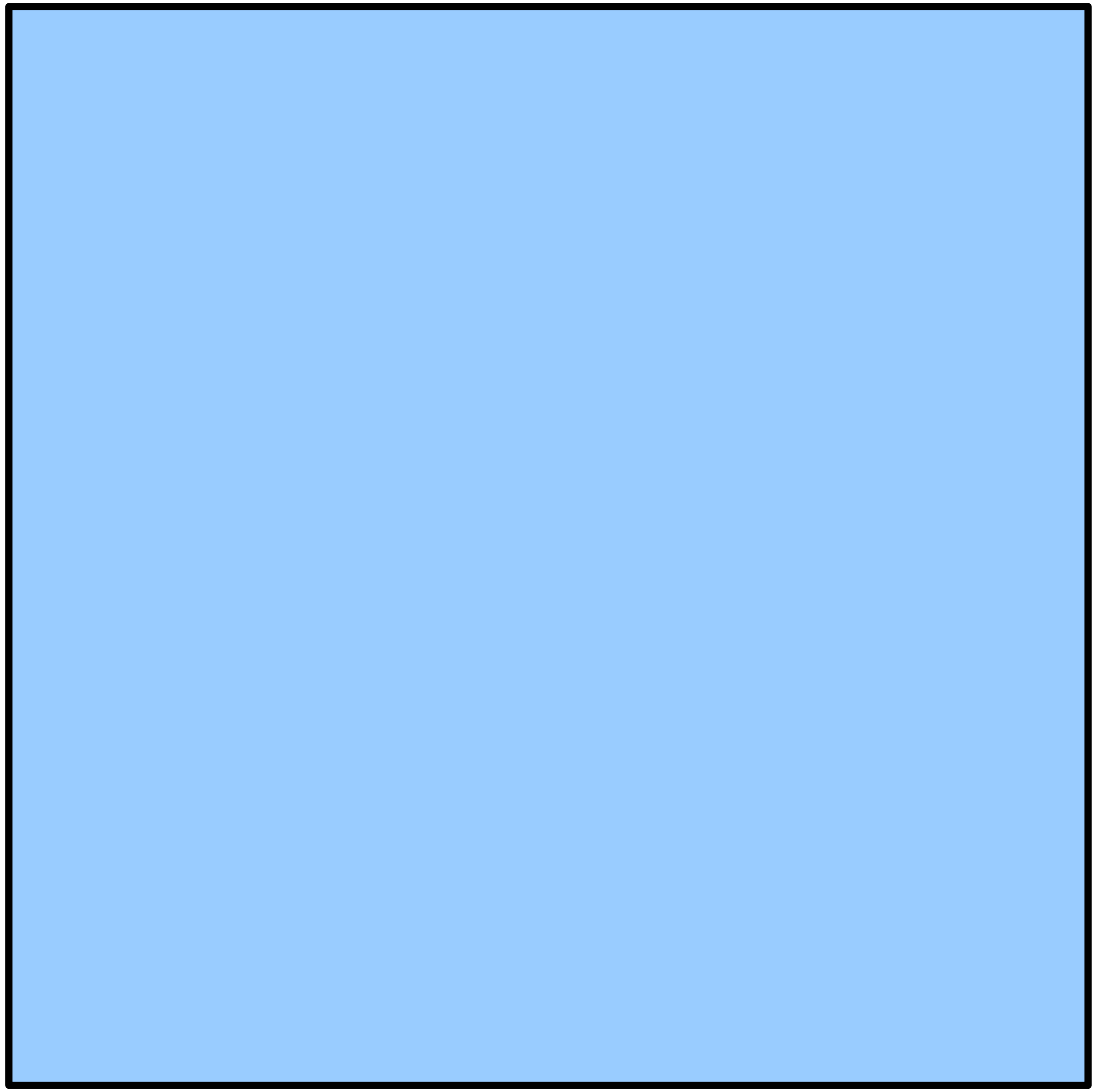

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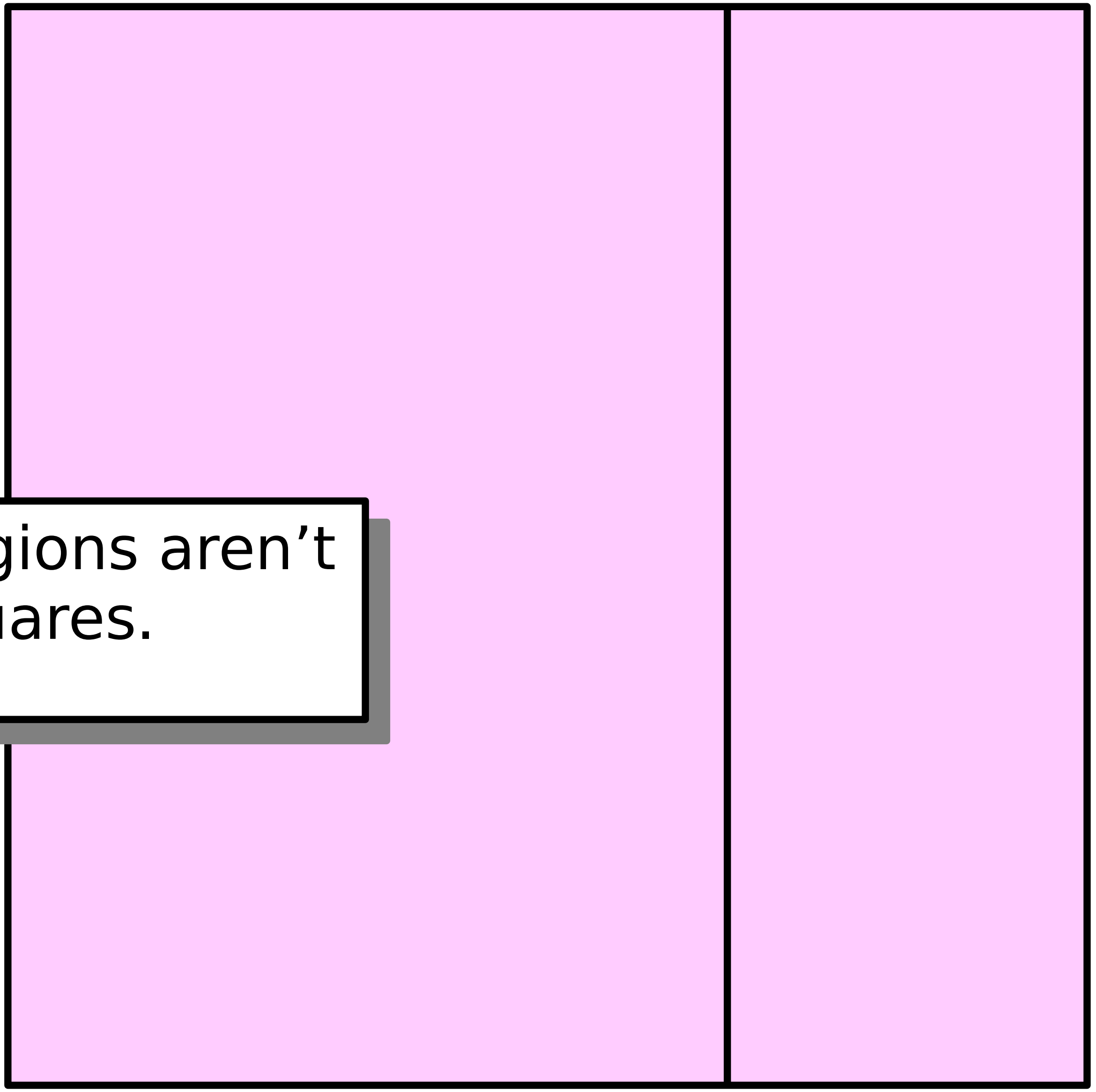
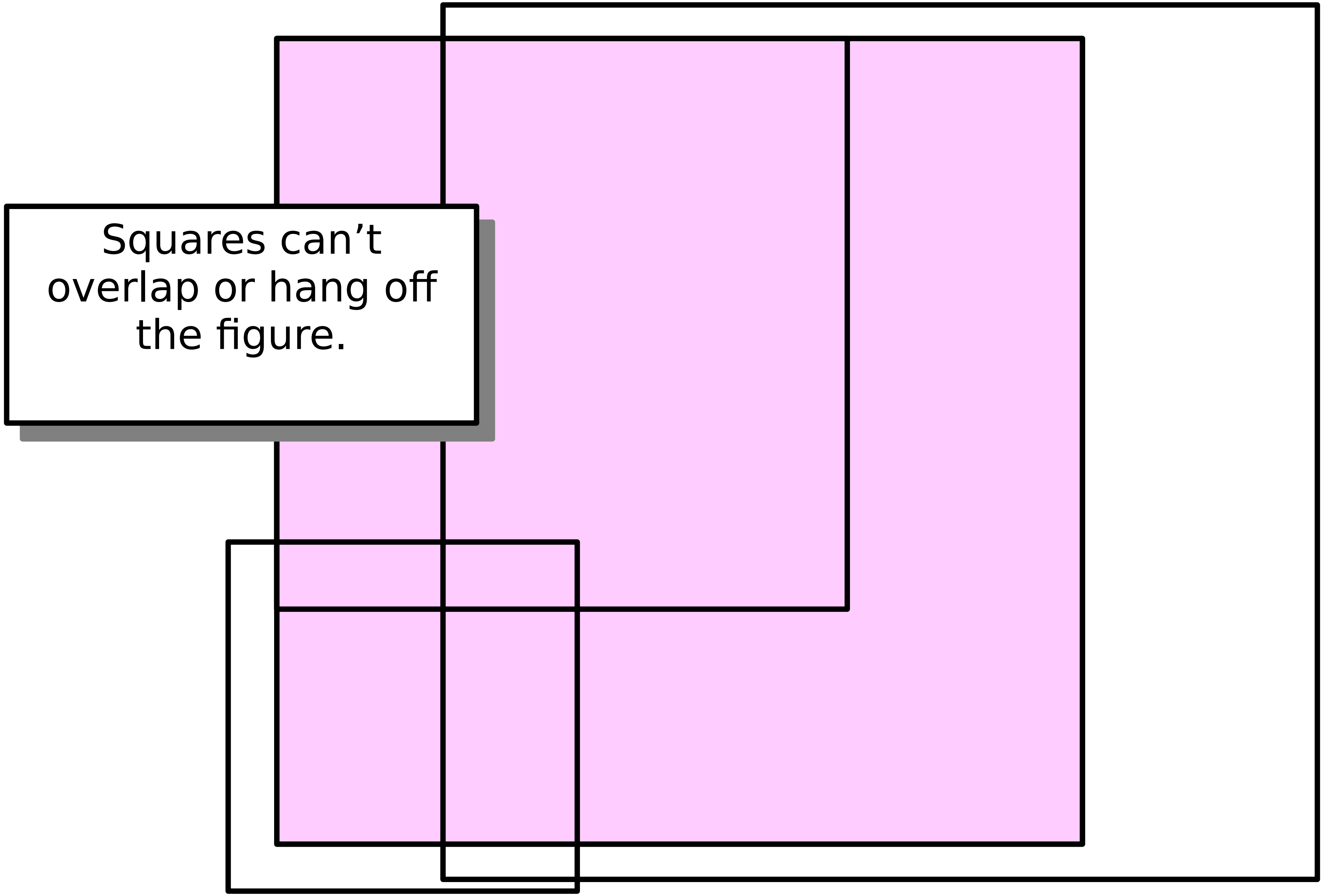


These regions aren’t squares.

Subdividing a Square



For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

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1

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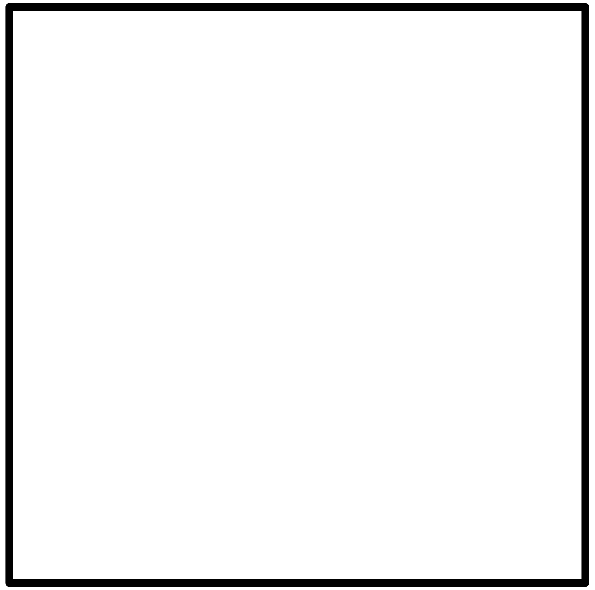
Each of the original corners needs to be covered by a corner of the new smaller squares.



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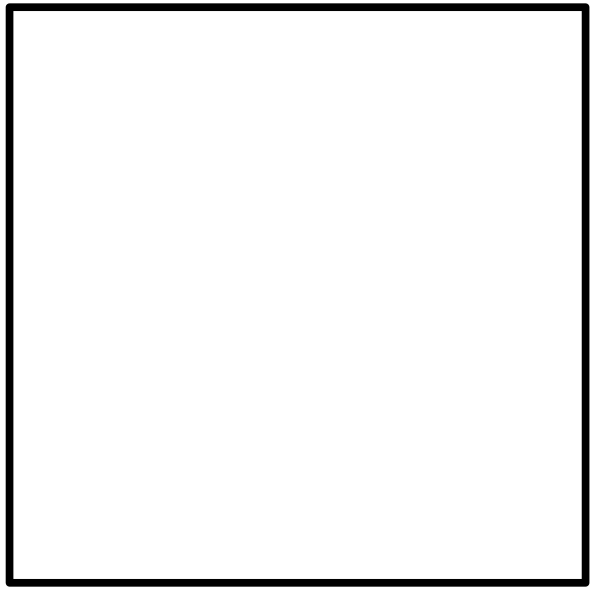

corners: 4

squares: <4

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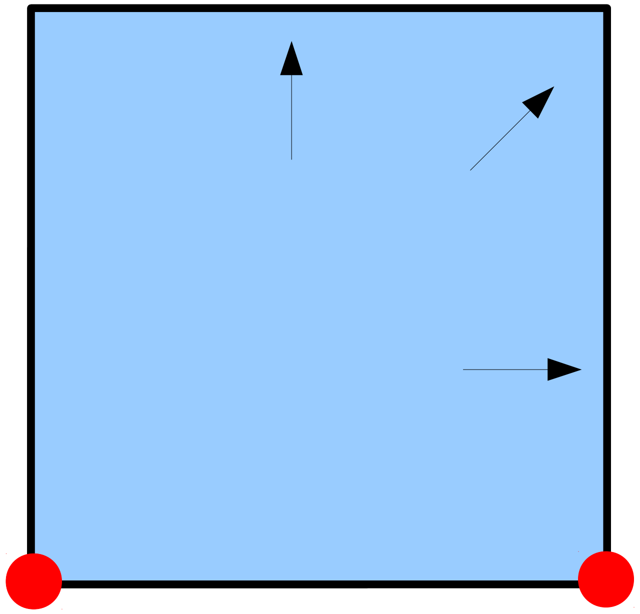
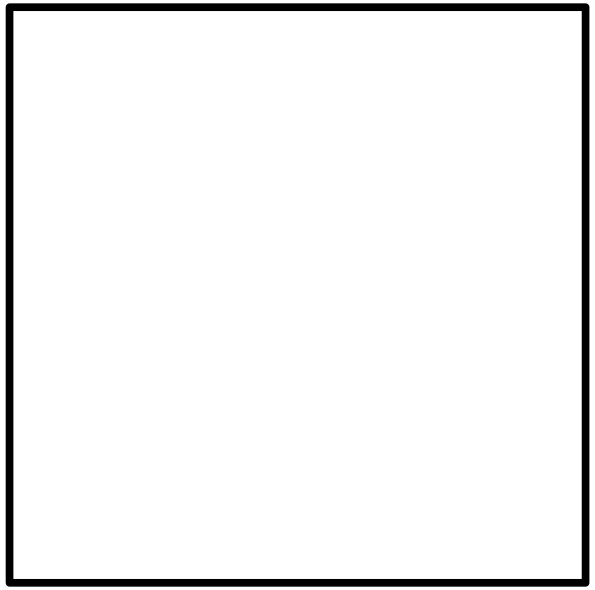


By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1	2
4	3

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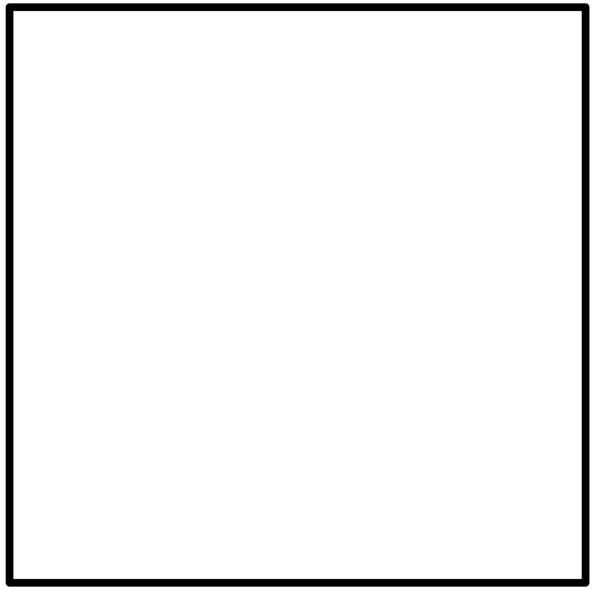
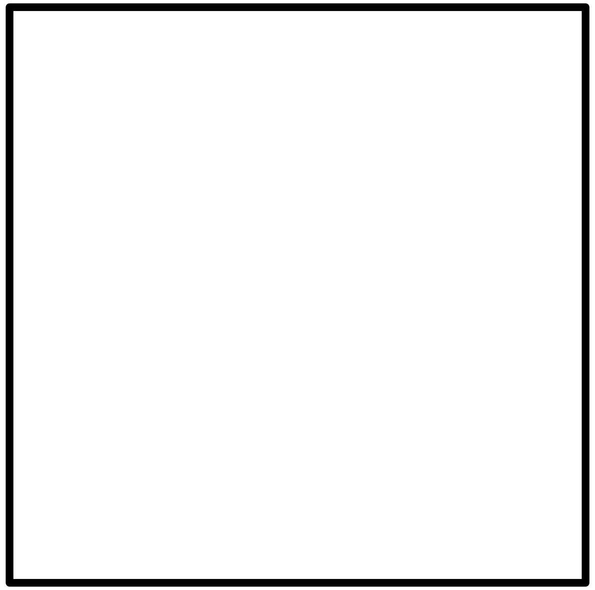


corners: 4

squares: 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

At least one square cannot be covering *any* of the original corners



corners: 4

squares: 5

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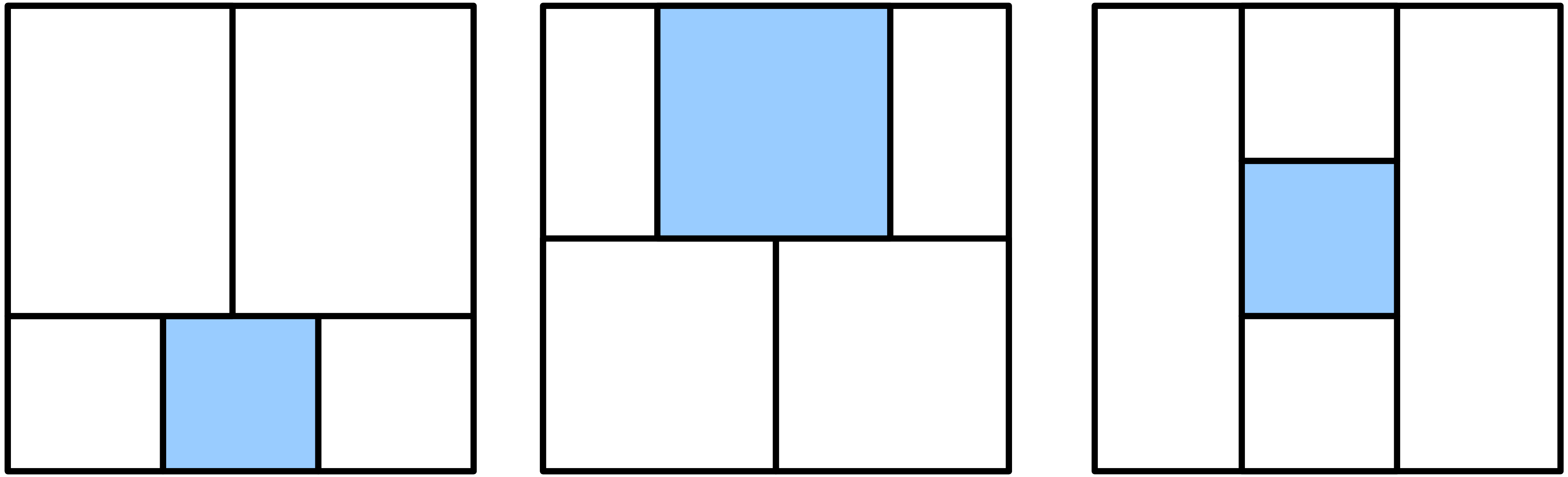

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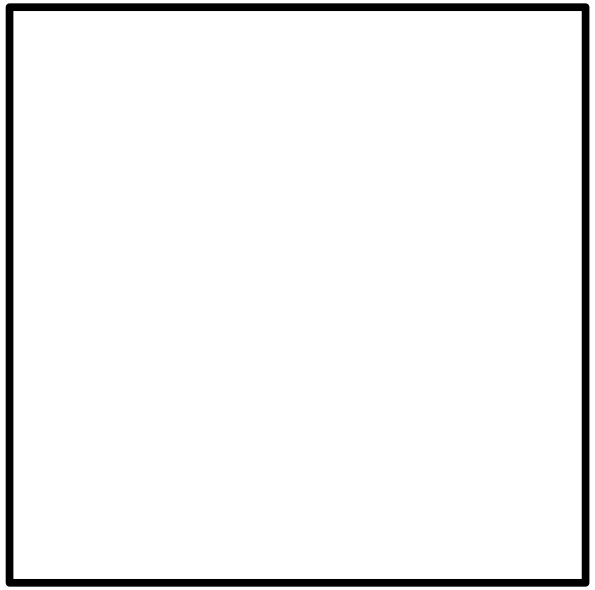

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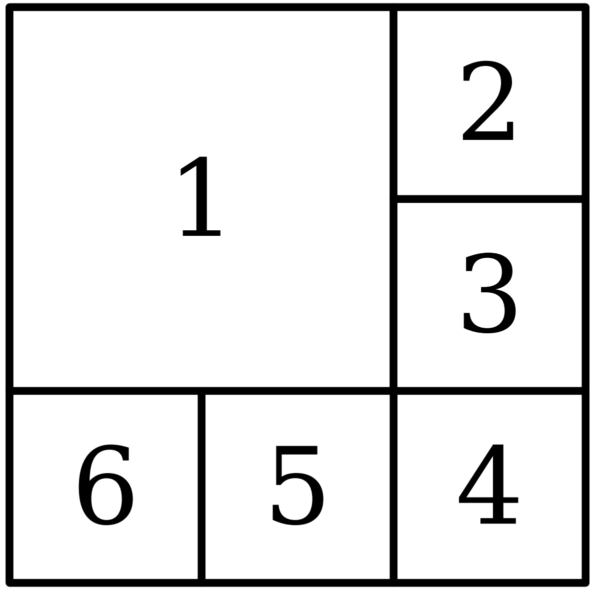

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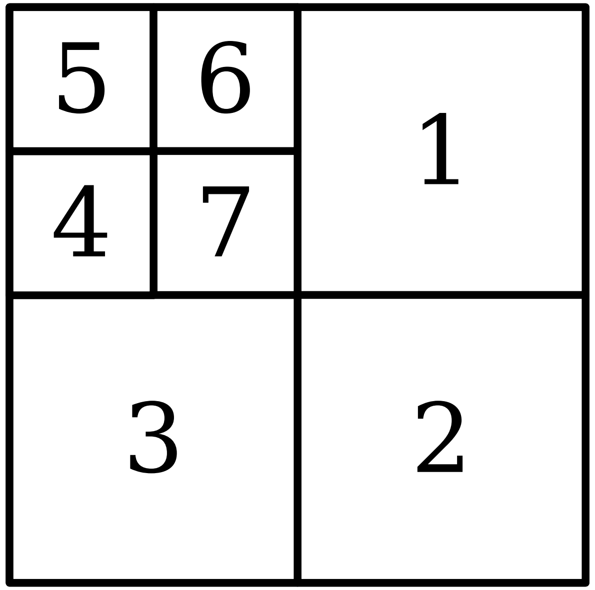

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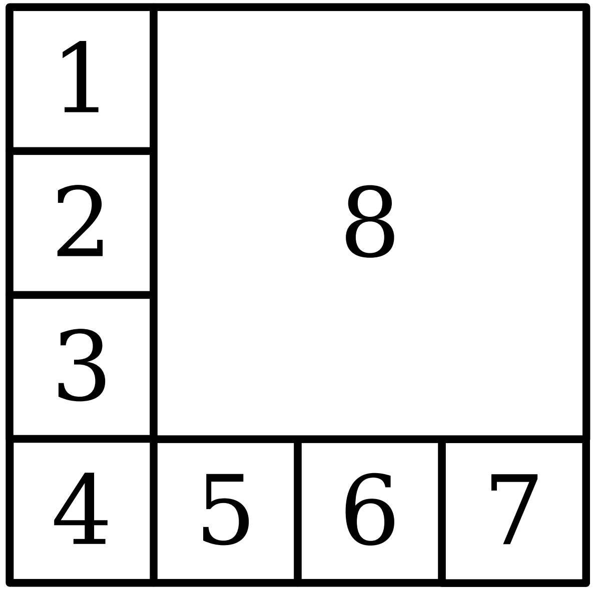

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

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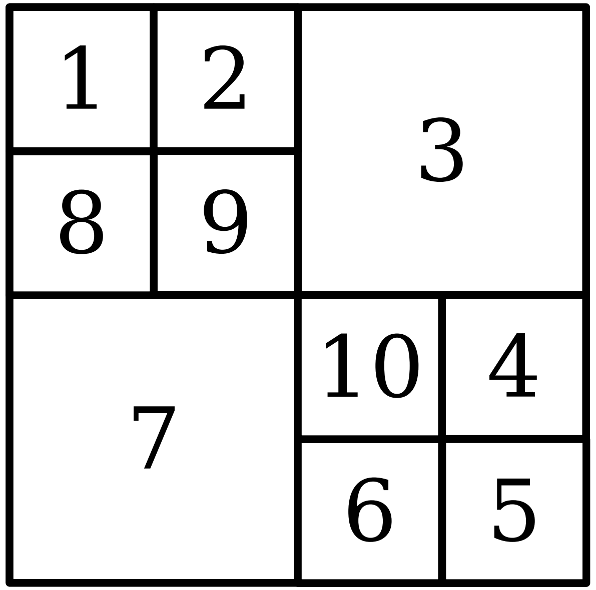

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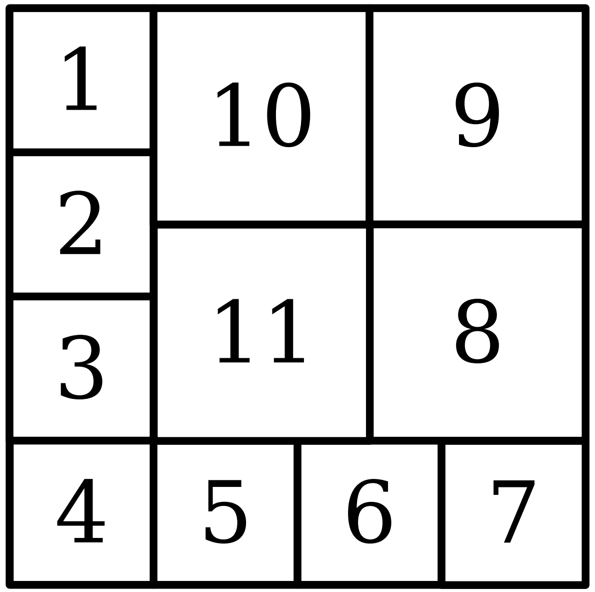

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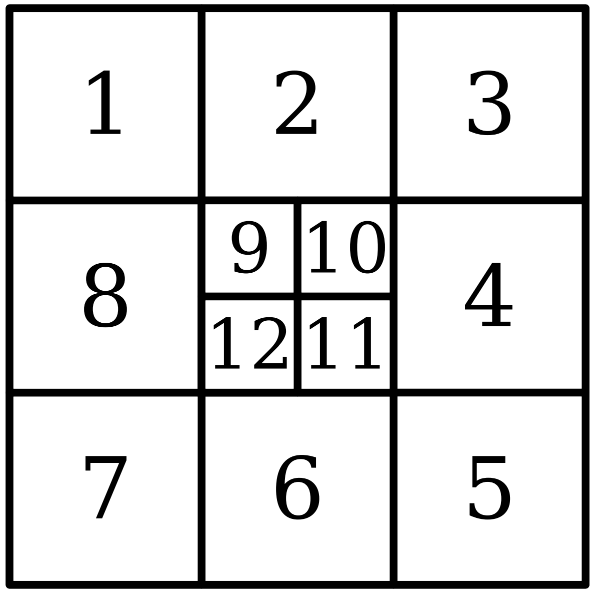

An Insight

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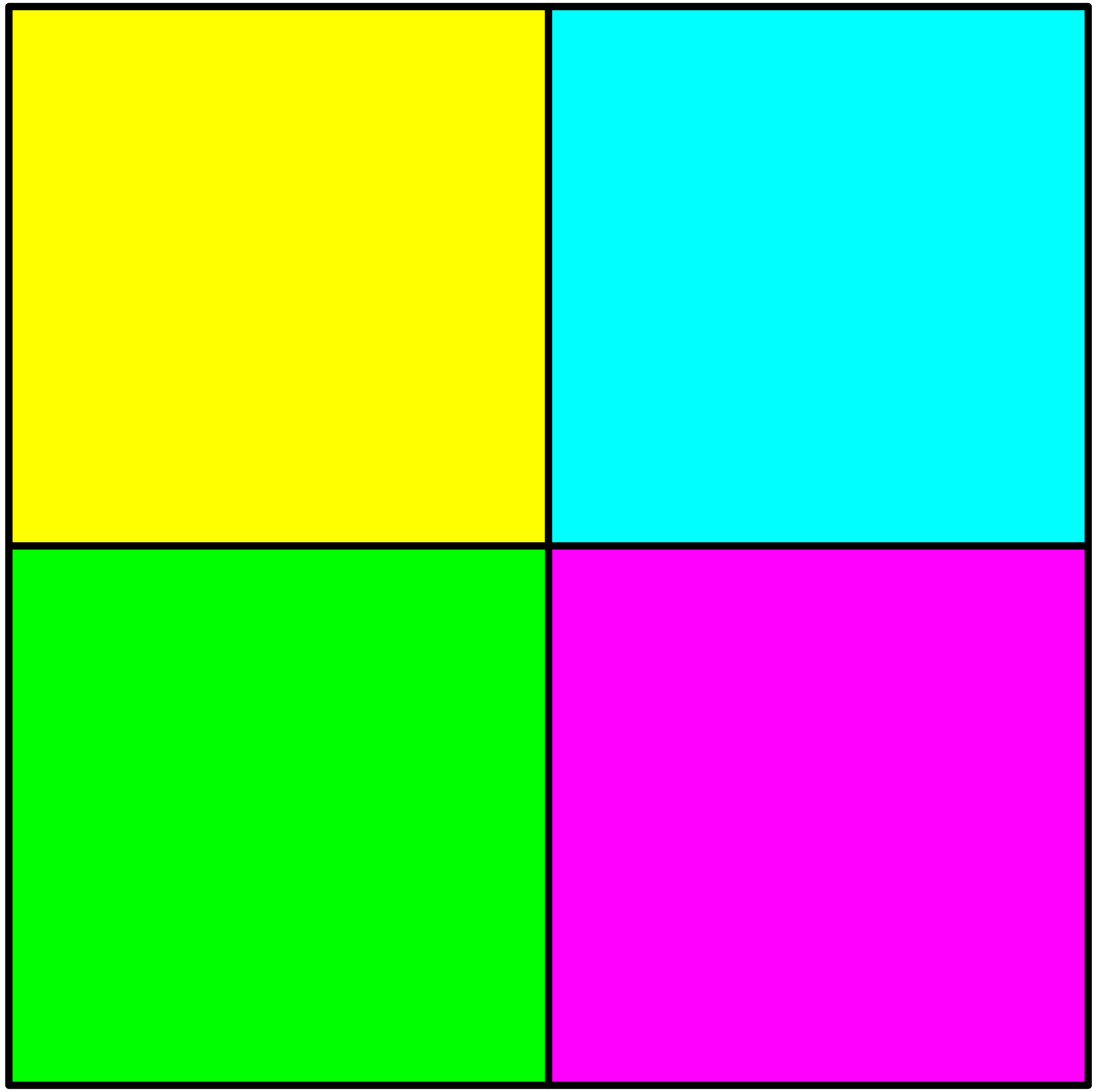

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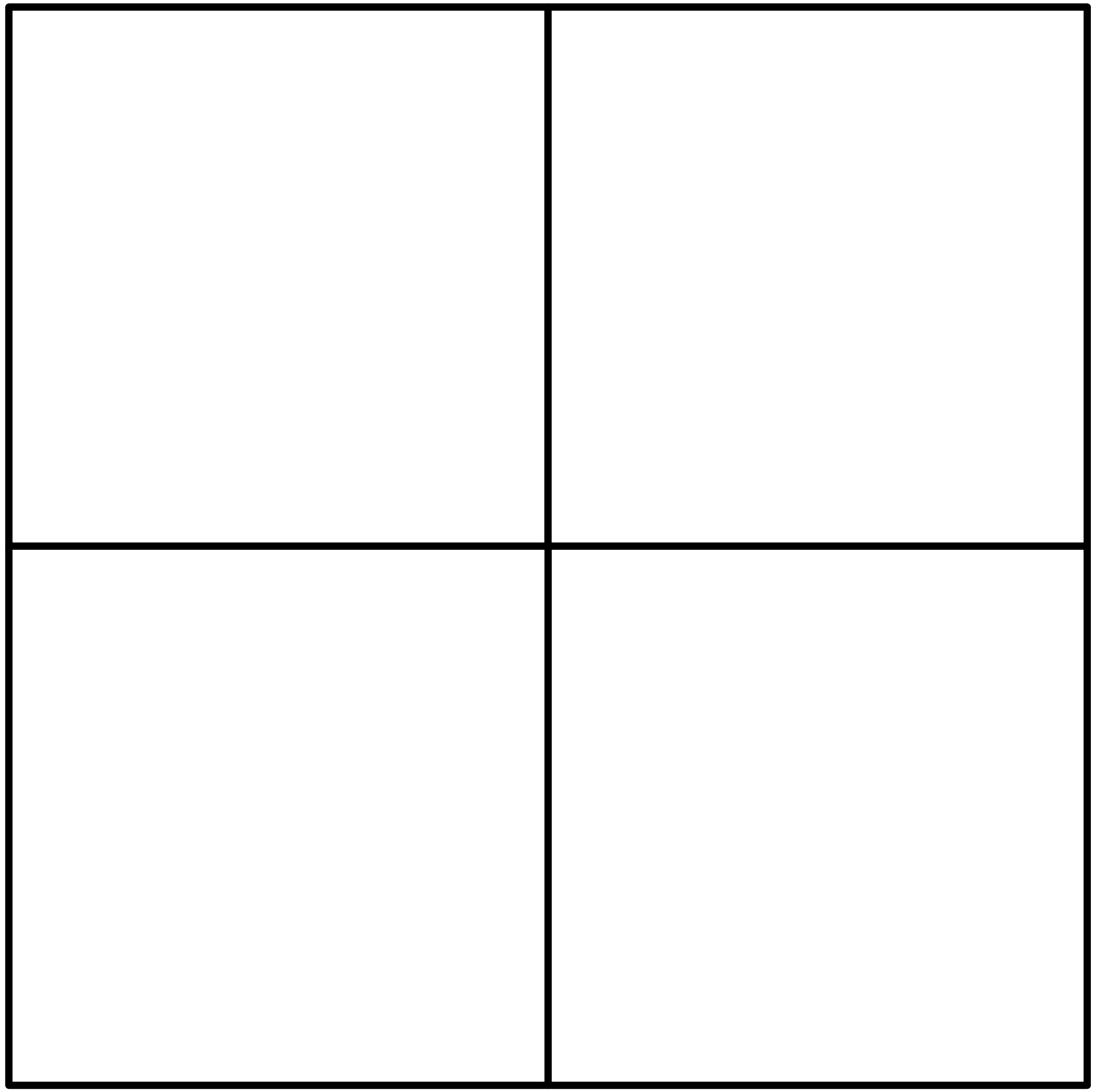

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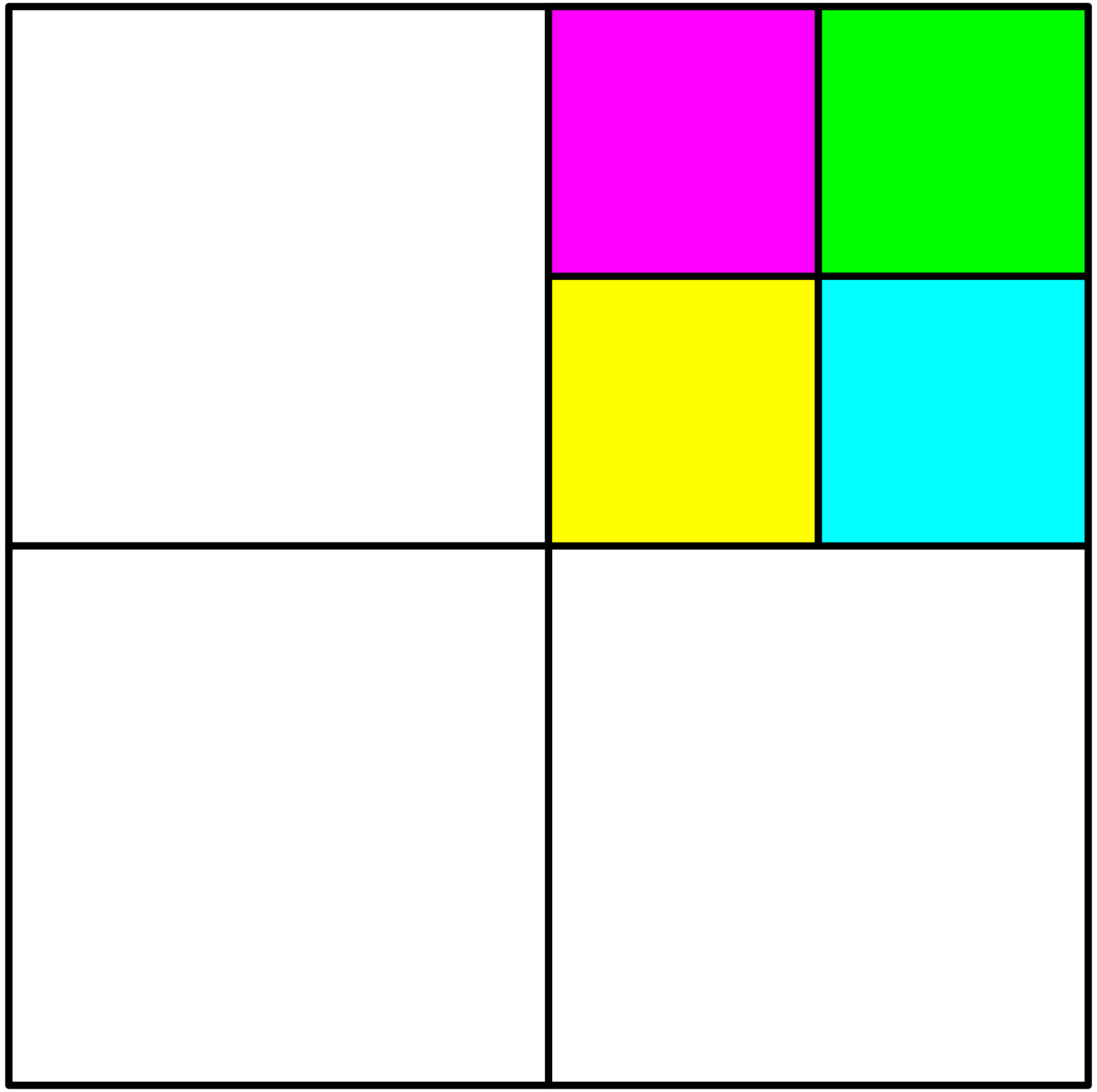

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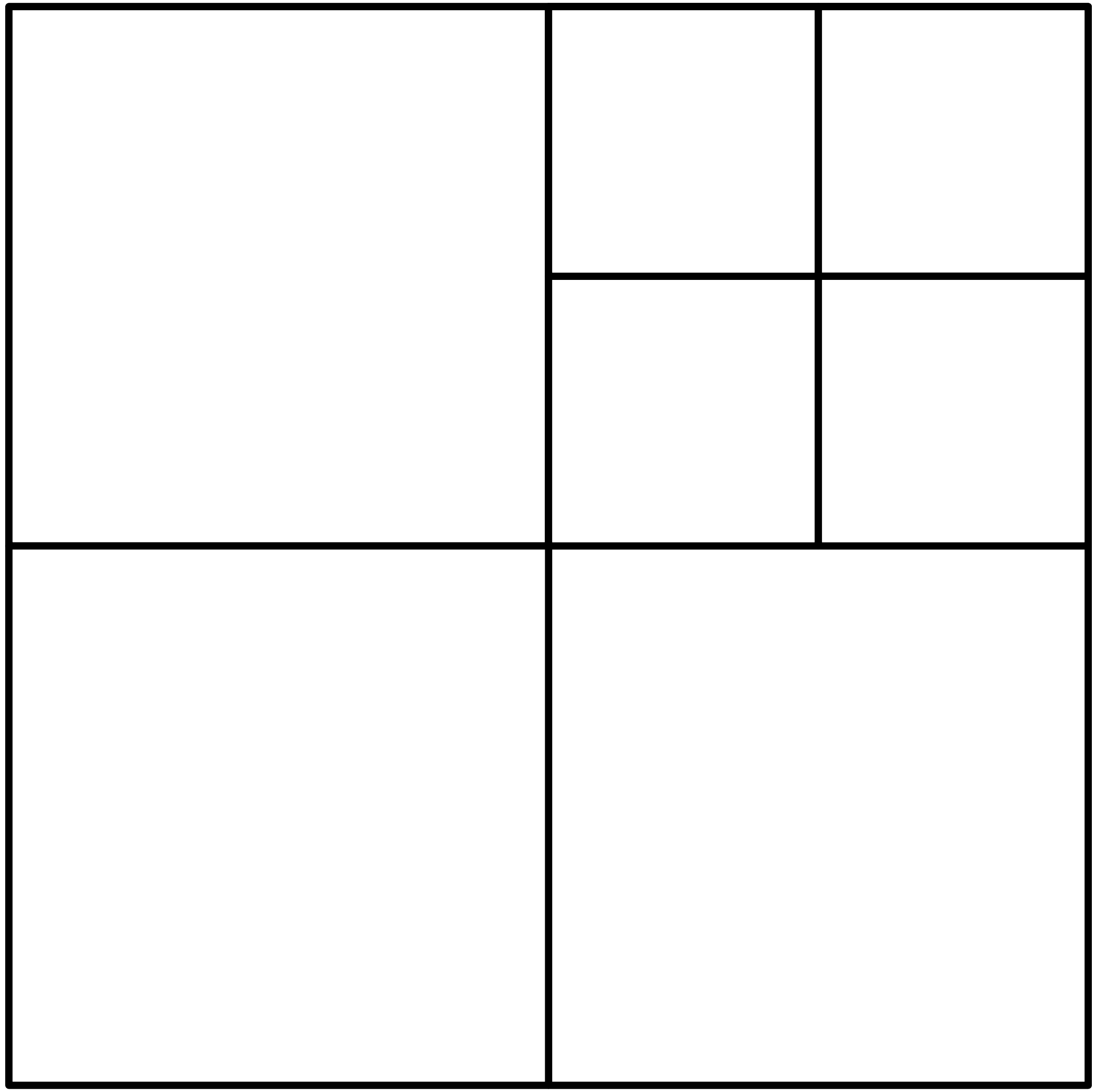

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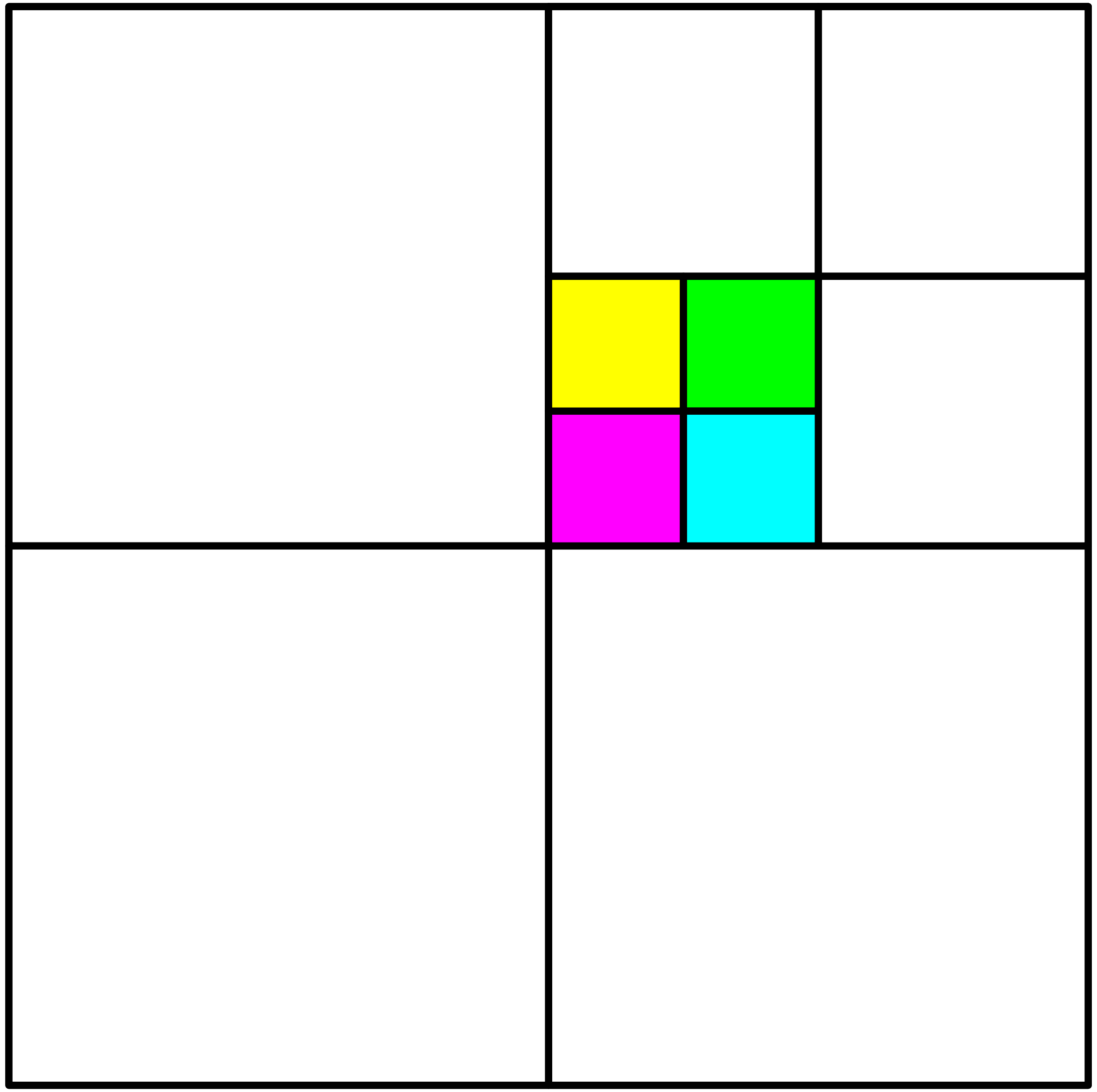

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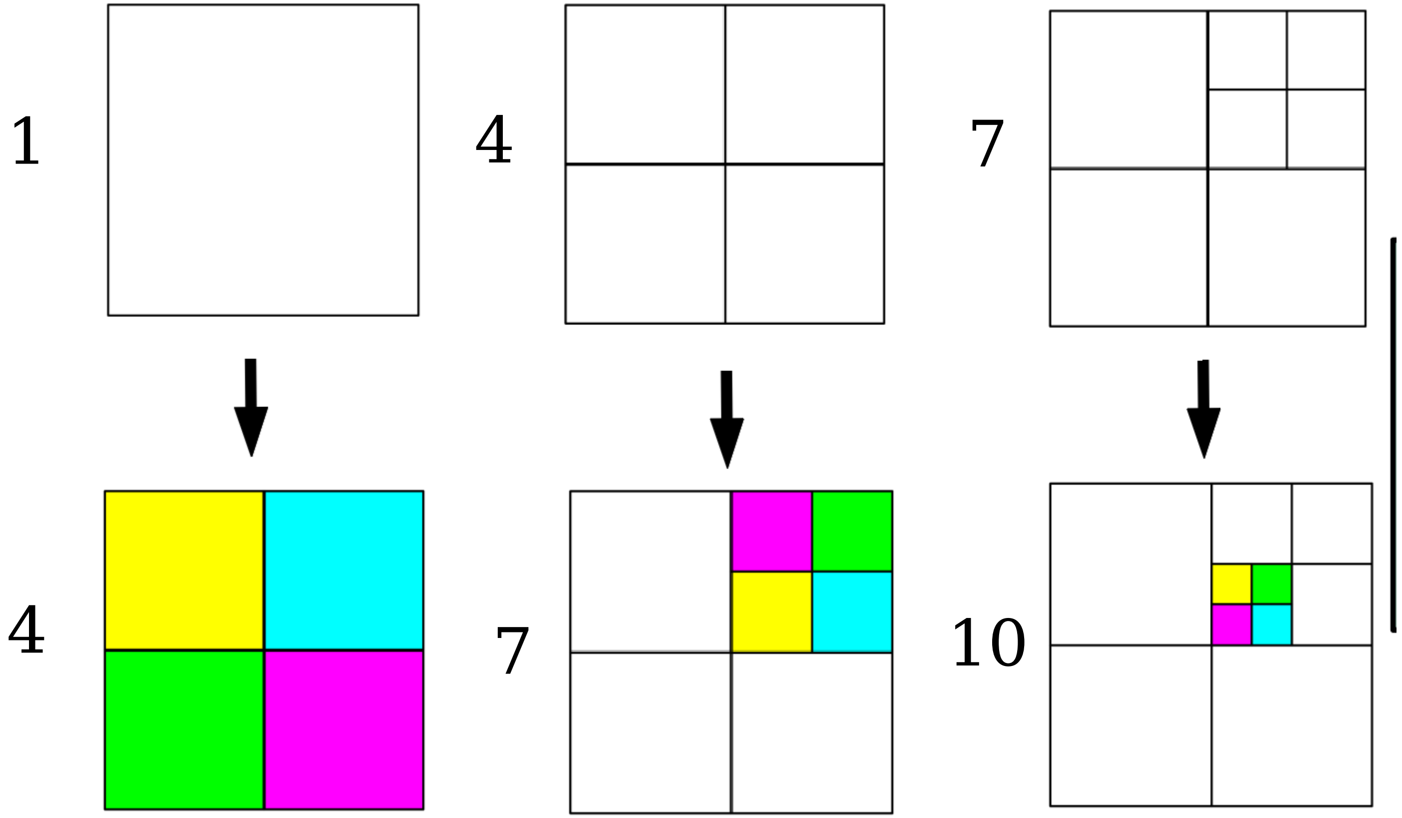

Quick check: We know we can do 1, 4, 7, 10. Based on this sequence, what are the next 3 sizes we know we can do? Go to PollEv.com/cs103spr25

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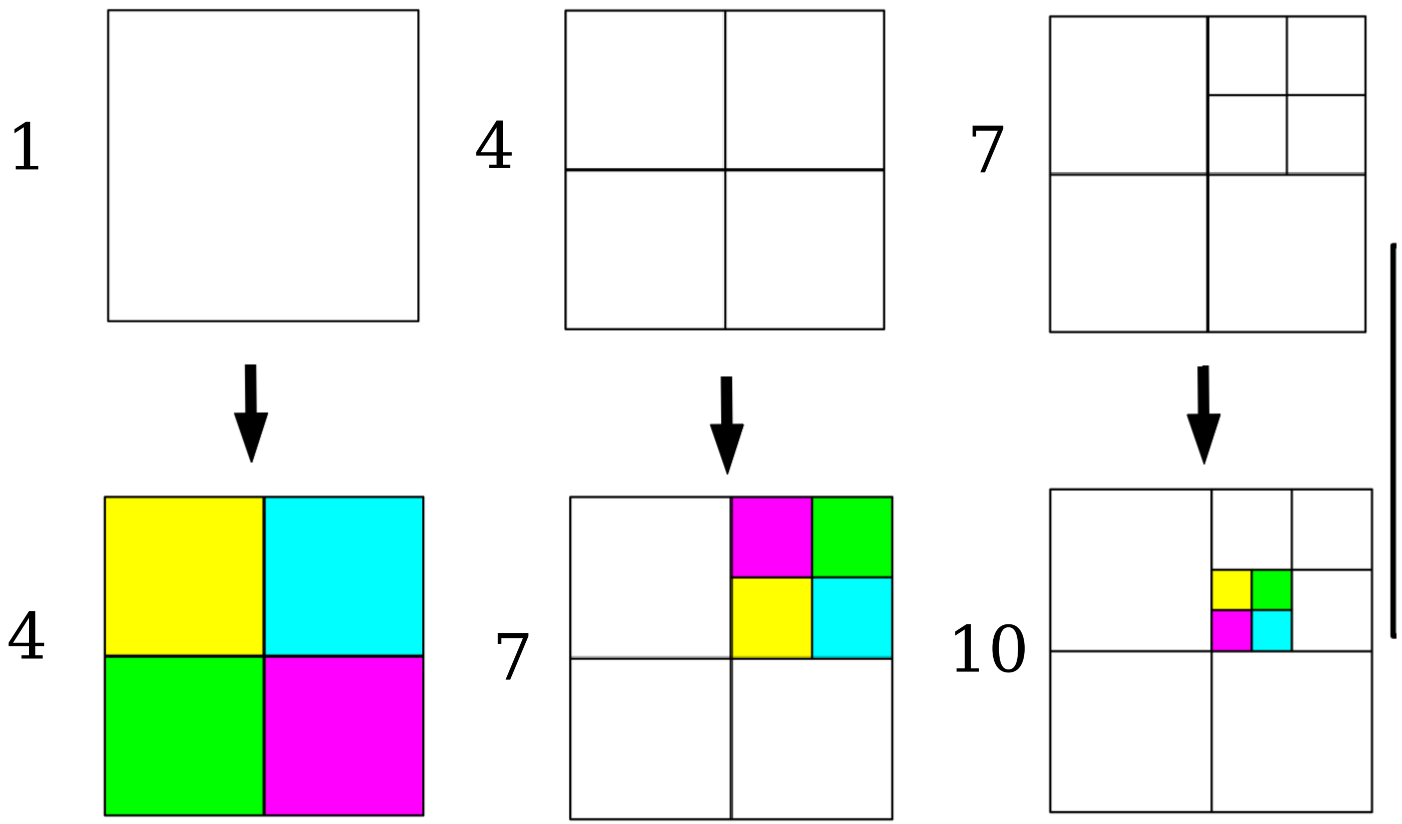

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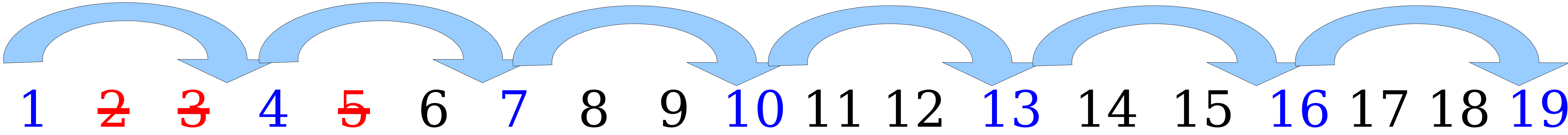

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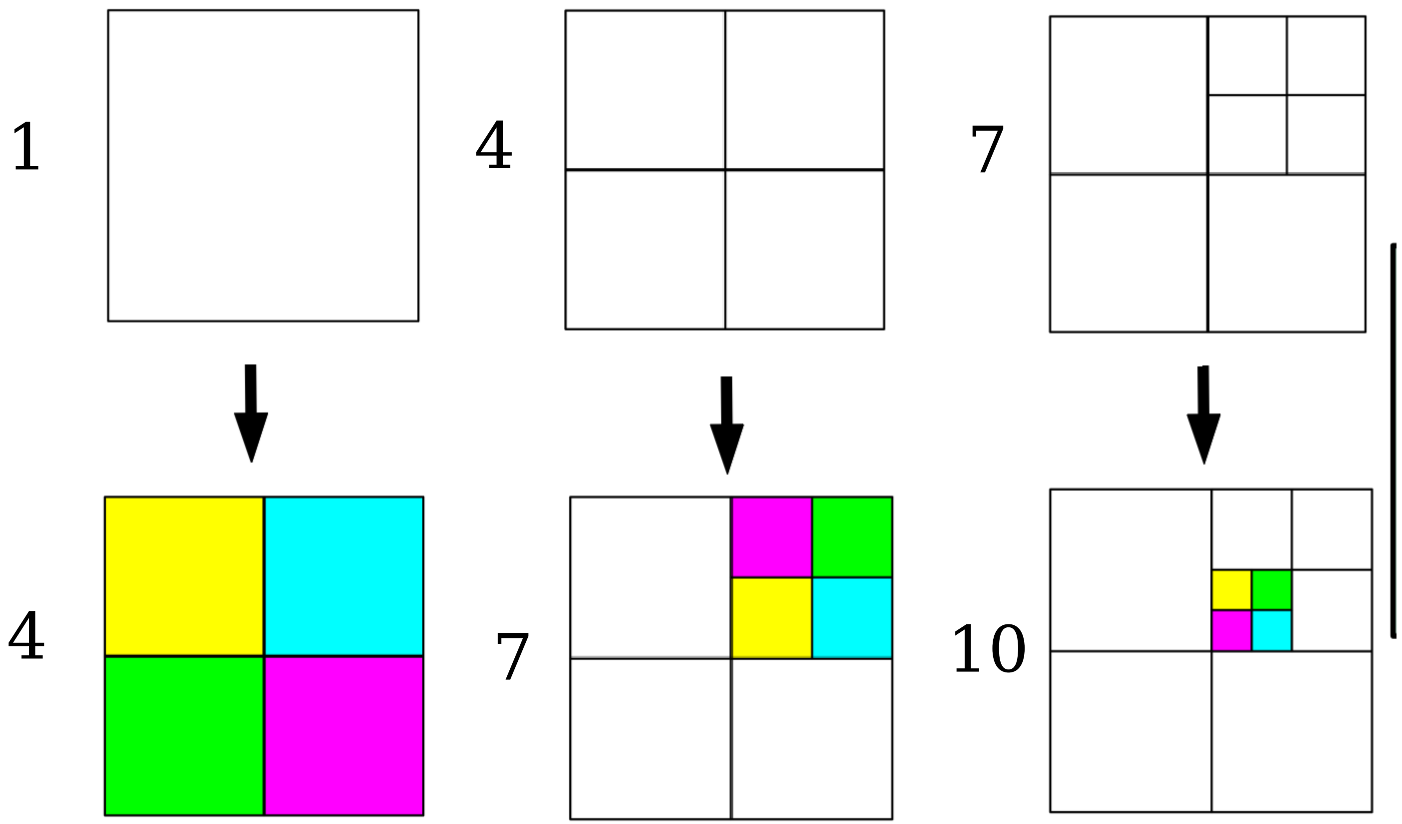
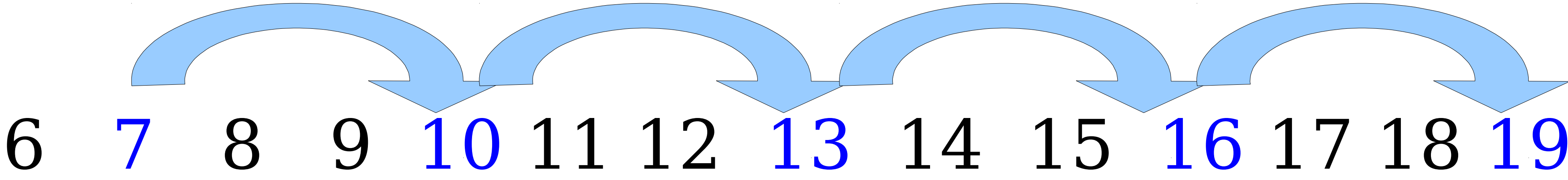


Follow-up: How could we get coverage of all numbers greater than 6, not just every third one? Go to PollEv.com/cs103spr25



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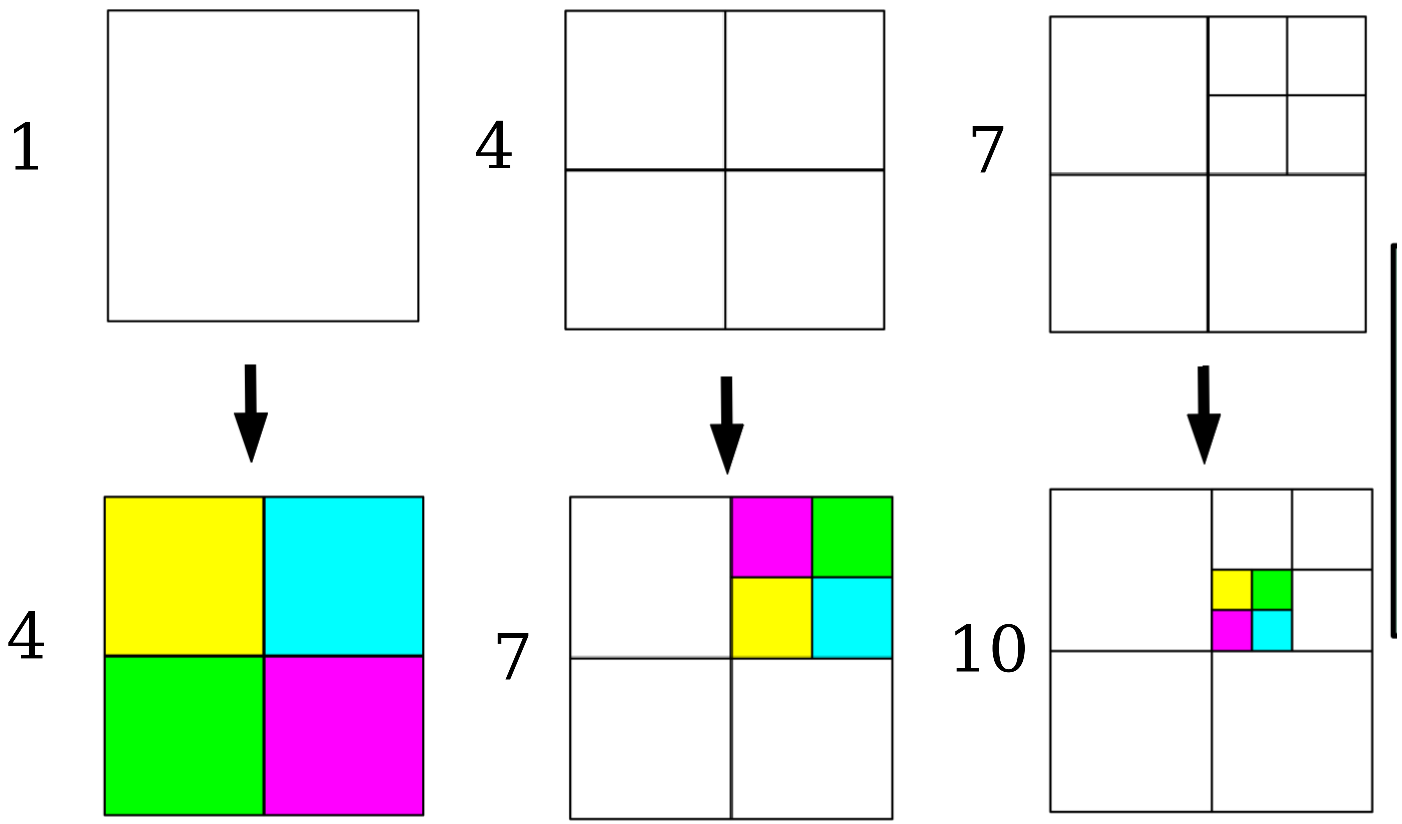

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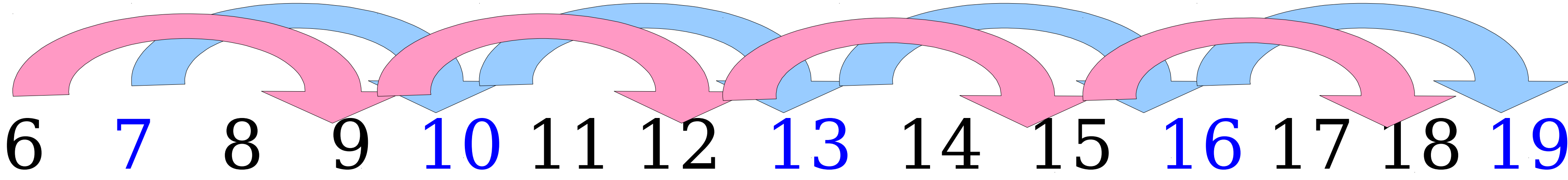

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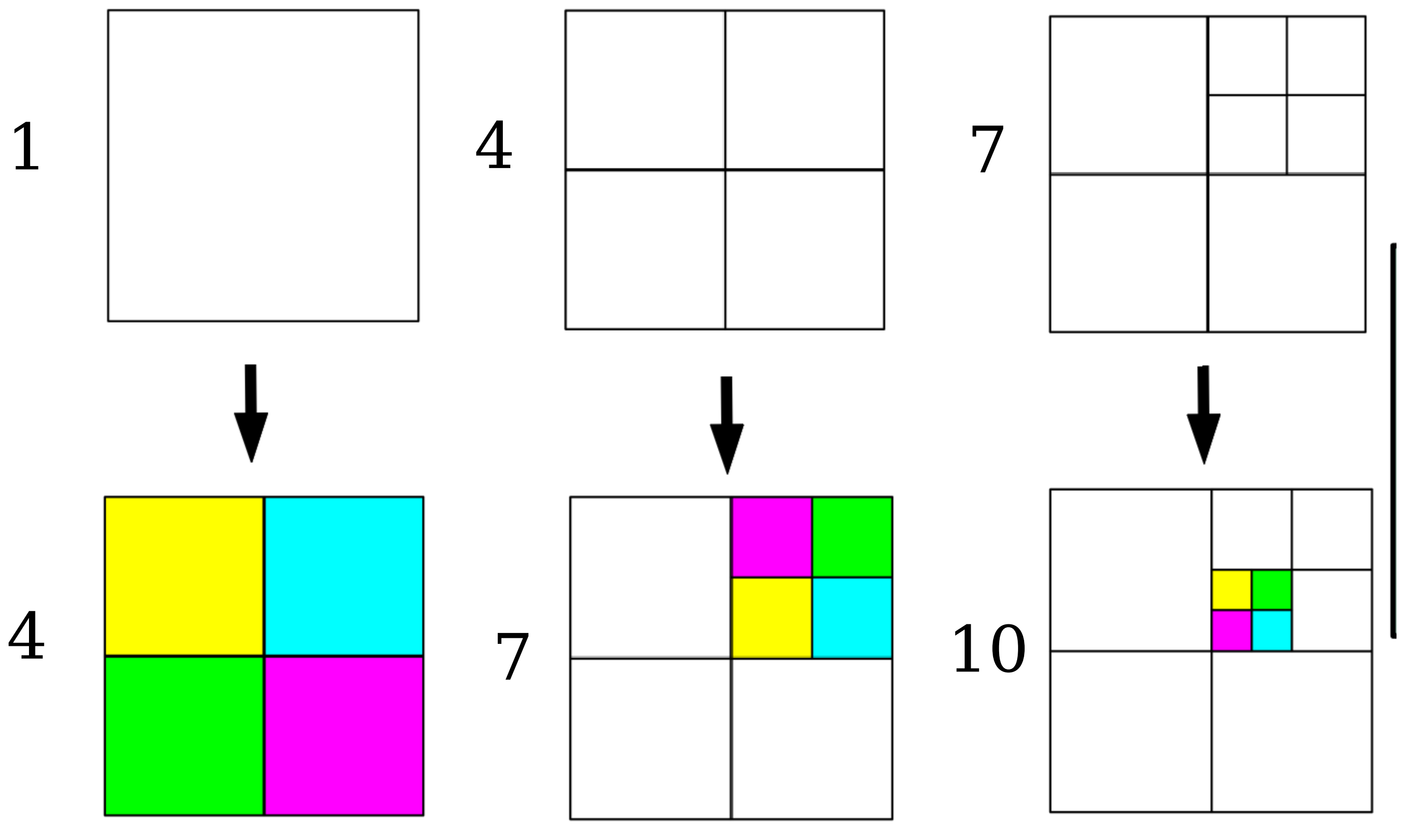

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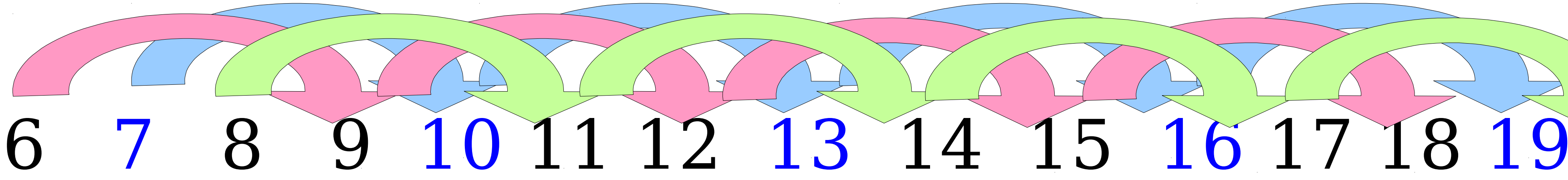

An Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

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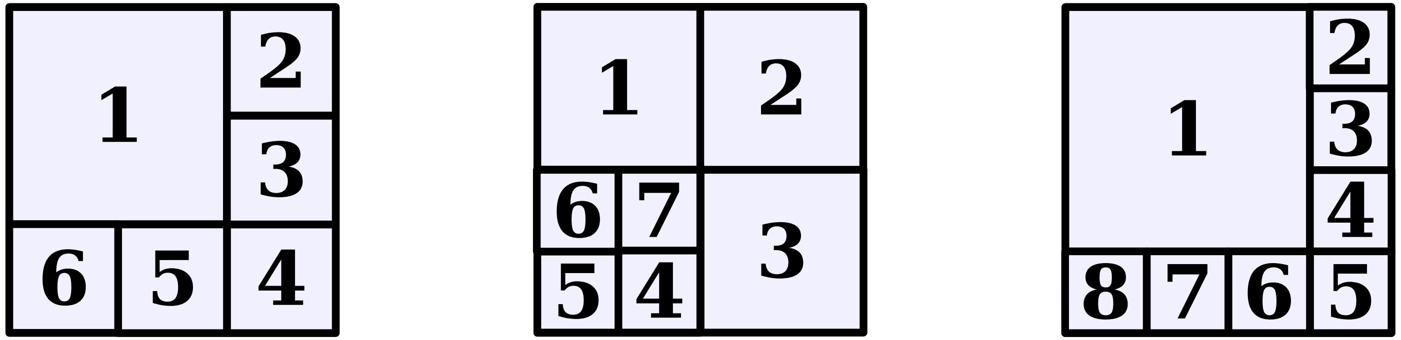

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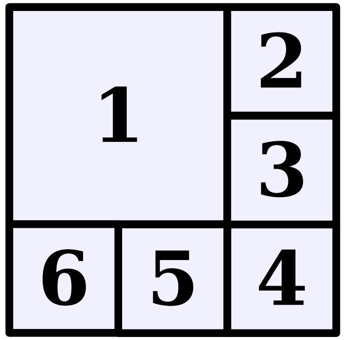

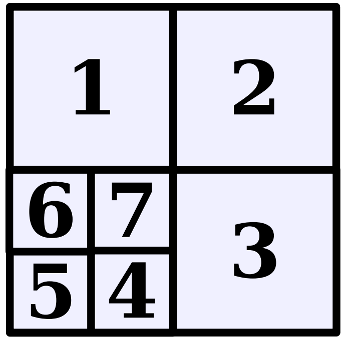

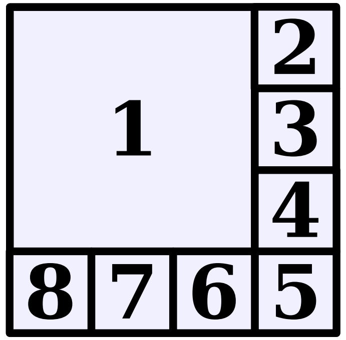

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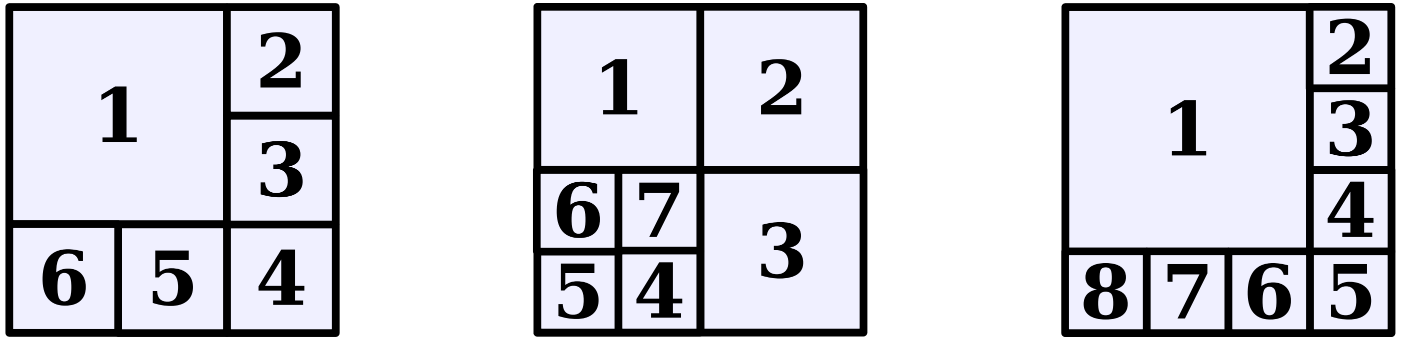

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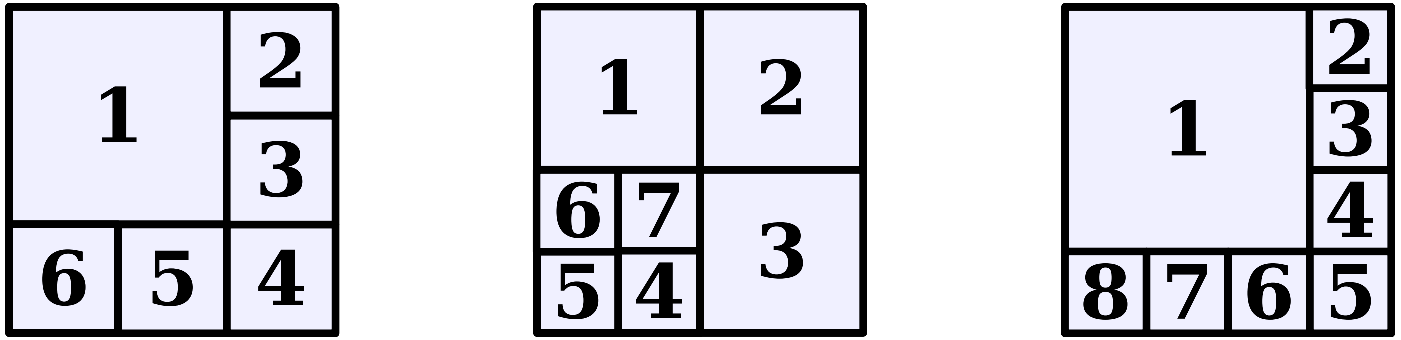

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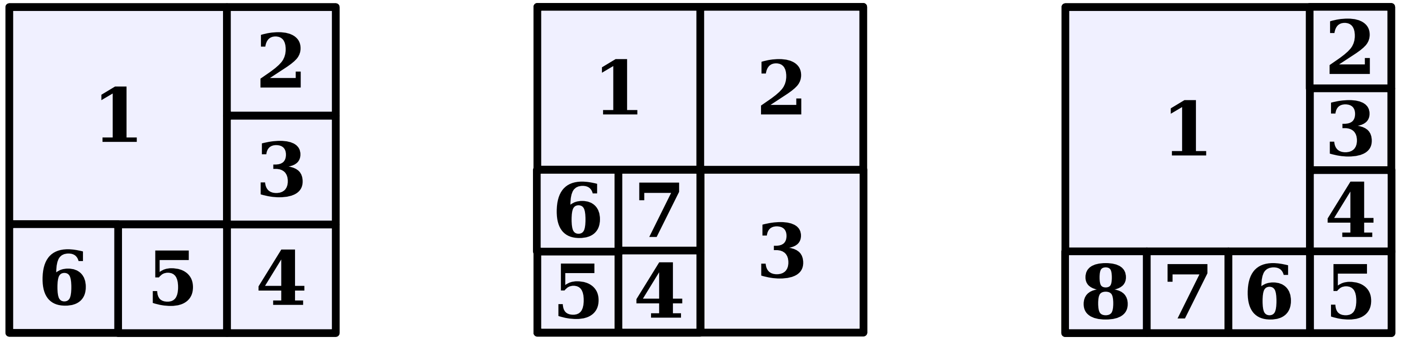

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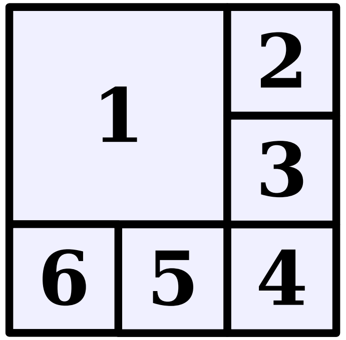

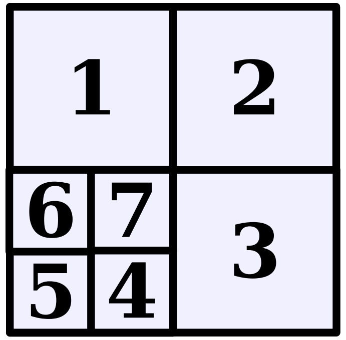

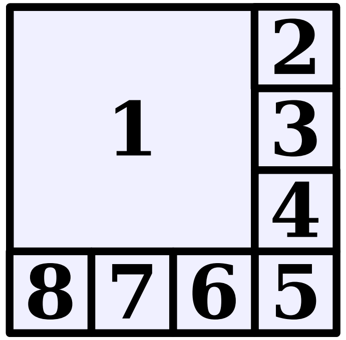

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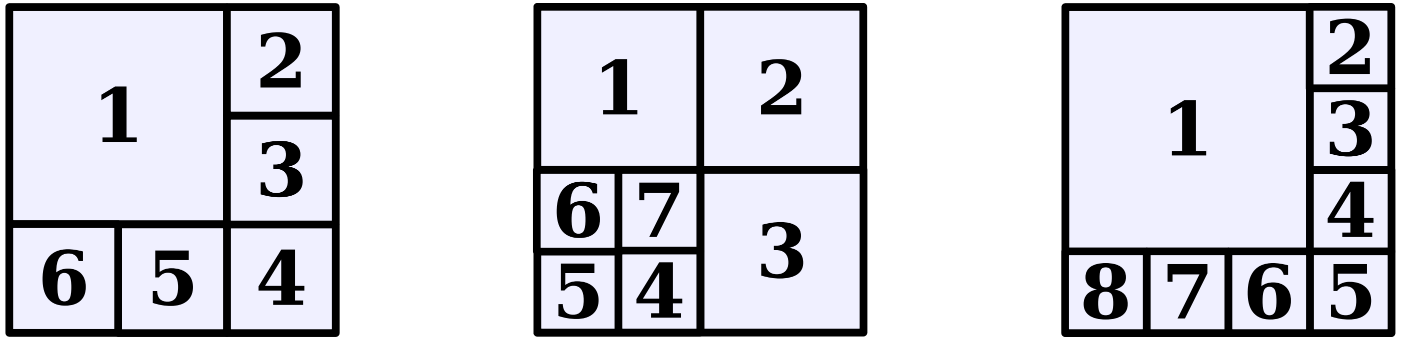

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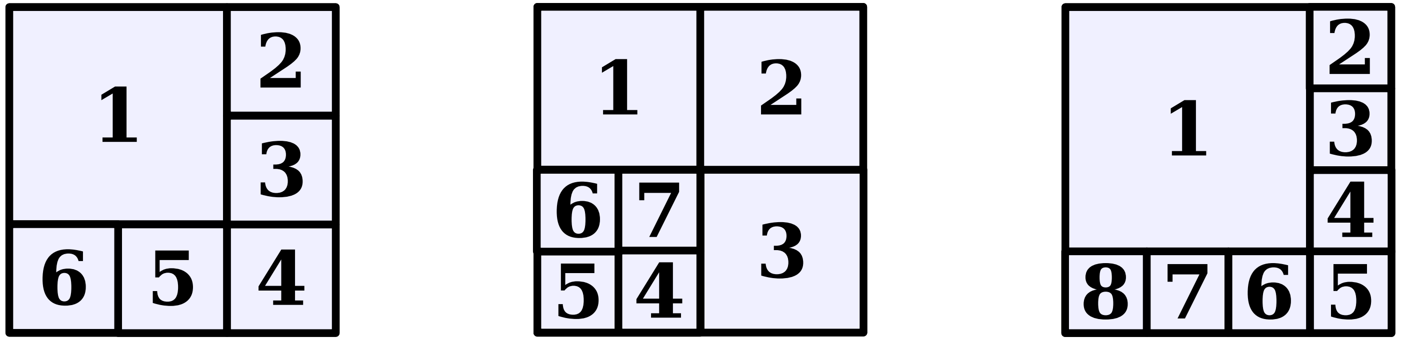

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Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don’t need to separately prove you’ve covered all the options. We trust you. 😃

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

Complete Induction

Let P be some predicate. The ***principle of complete induction*** states that if

$P(0)$ is true

If it starts true…

and

…and it stays true…

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

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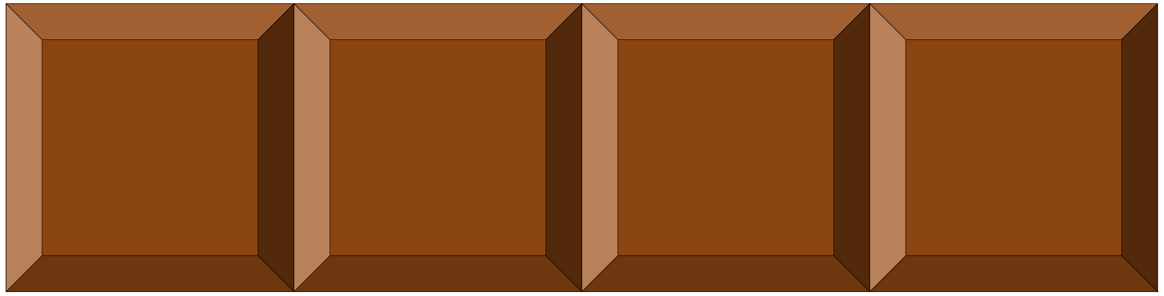
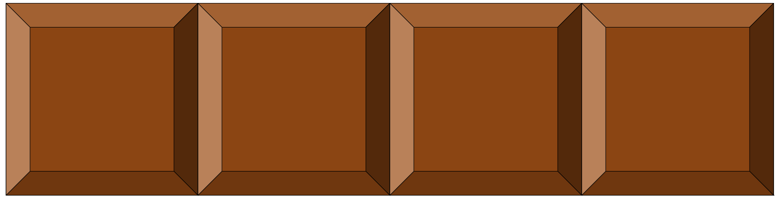
An Example: ***Eating a Chocolate Bar***



Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?



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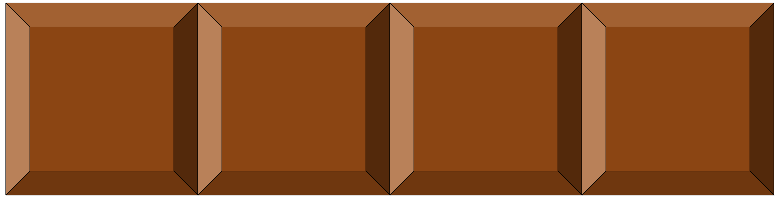

1×1: (1)

1×2: (1, 1), (2)

1×3:

Eating a Chocolate Bar

Quick check: What is the answer for 1x3?
Go to
PollEv.com/cs103spr25

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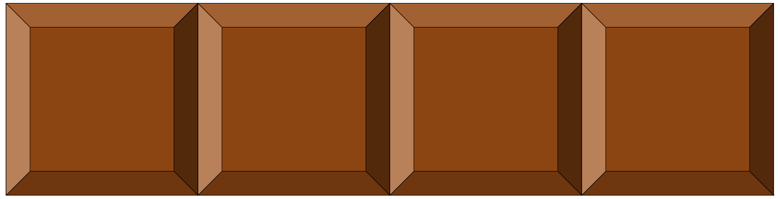

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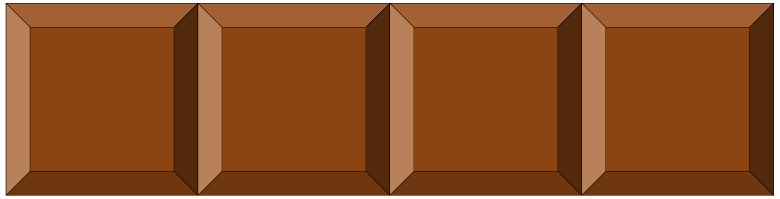

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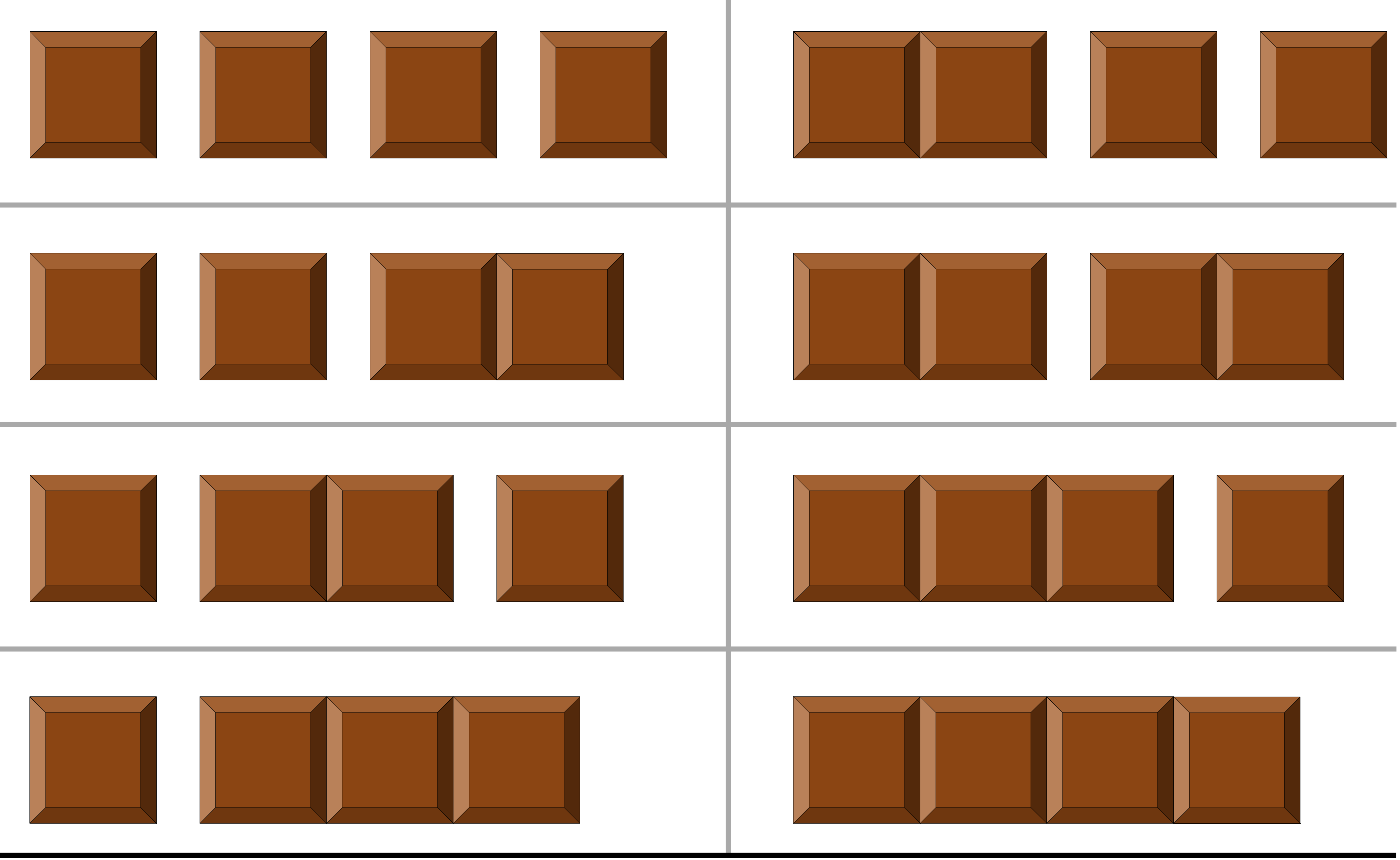

There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1 × 1 chocolate bar,
  - 2 ways to eat a 1 × 2 chocolate bar,
  - 4 ways to eat a 1 × 3 chocolate bar, and
  - 8 ways to eat a 1 × 4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

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Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \ldots + 2^2 + 2^1 + 2^0$$

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Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \ldots + 2^2 + 2^1 + 2^0 \quad = \quad 1 + 2^k - 1 \quad = \quad 2^k.$$

Thus $P(k+1)$ holds, completing the induction.

Theorem: For any natural number $n \geq 1$, there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.

Proof: Let $P(n)$ be the statement "there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

As our base case, we prove $P(1)$, that there is exactly $2^{1-1} = 1$ way to eat a 1×1 chocolate bar from left to right. The only option here is to eat the entire chocolate bar at once, so there's just one way to eat it, as needed.

For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, …, and $P(k)$ are true. We need to show $P(k+1)$ is true, that there are exactly 2^k ways to eat a $1 \times (k+1)$ chocolate bar.

There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right. Since $1 \leq r \leq k$, we know that $1 \leq k+1-r \leq k$, so by our inductive hypothesis there are 2^{k-r} ways to eat the remainder.

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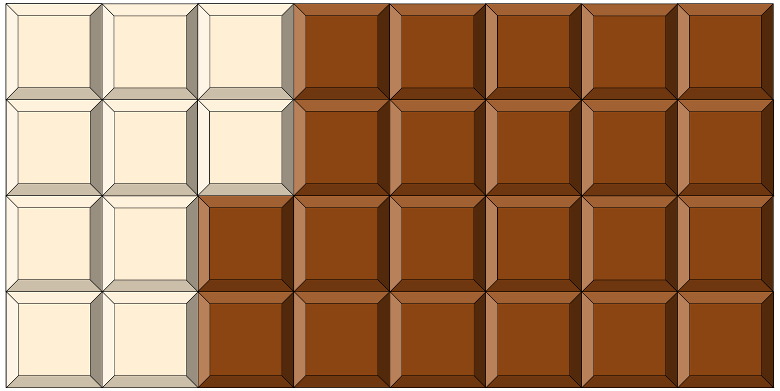
Thus $P(k+1)$ holds, completing the induction. ■

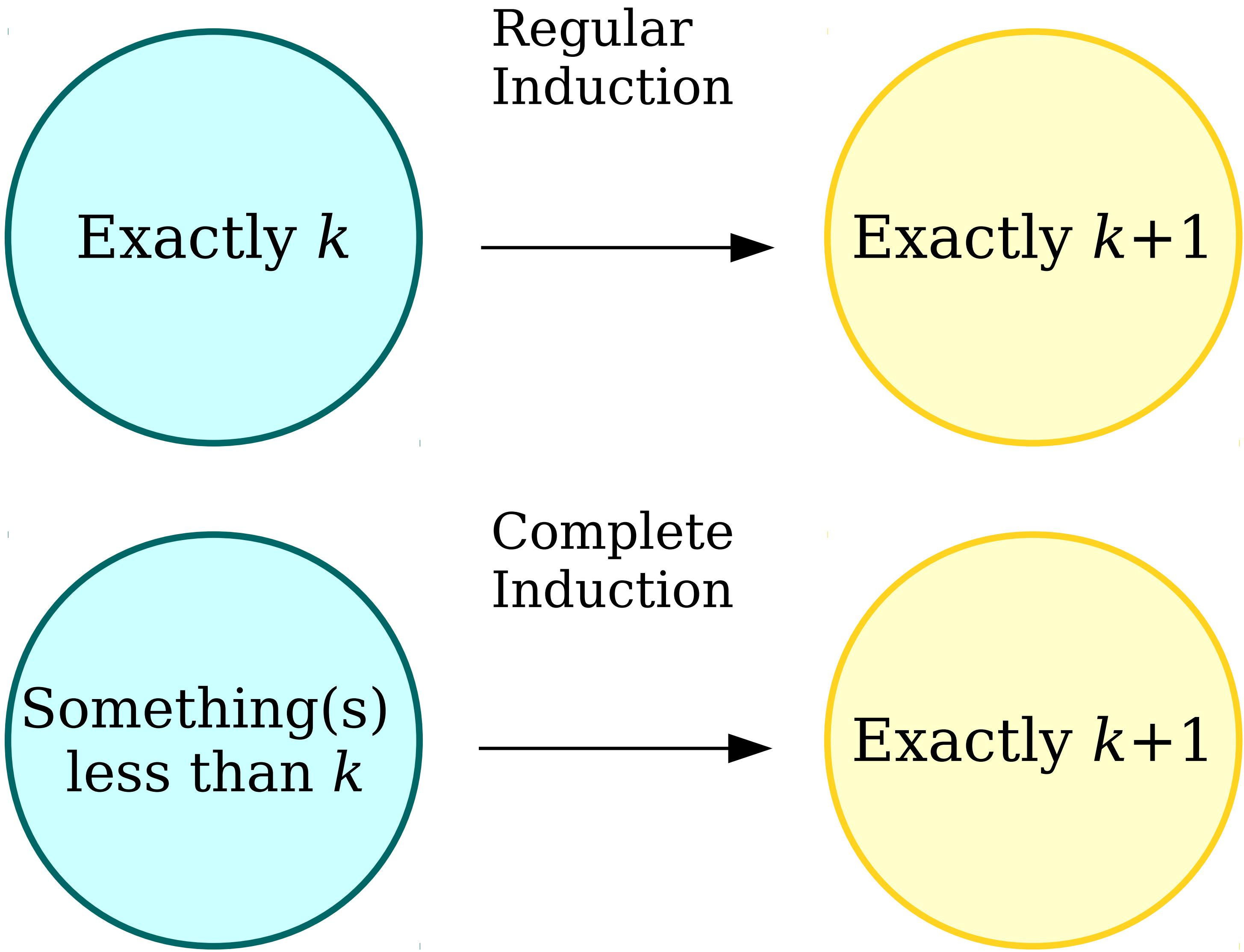
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?

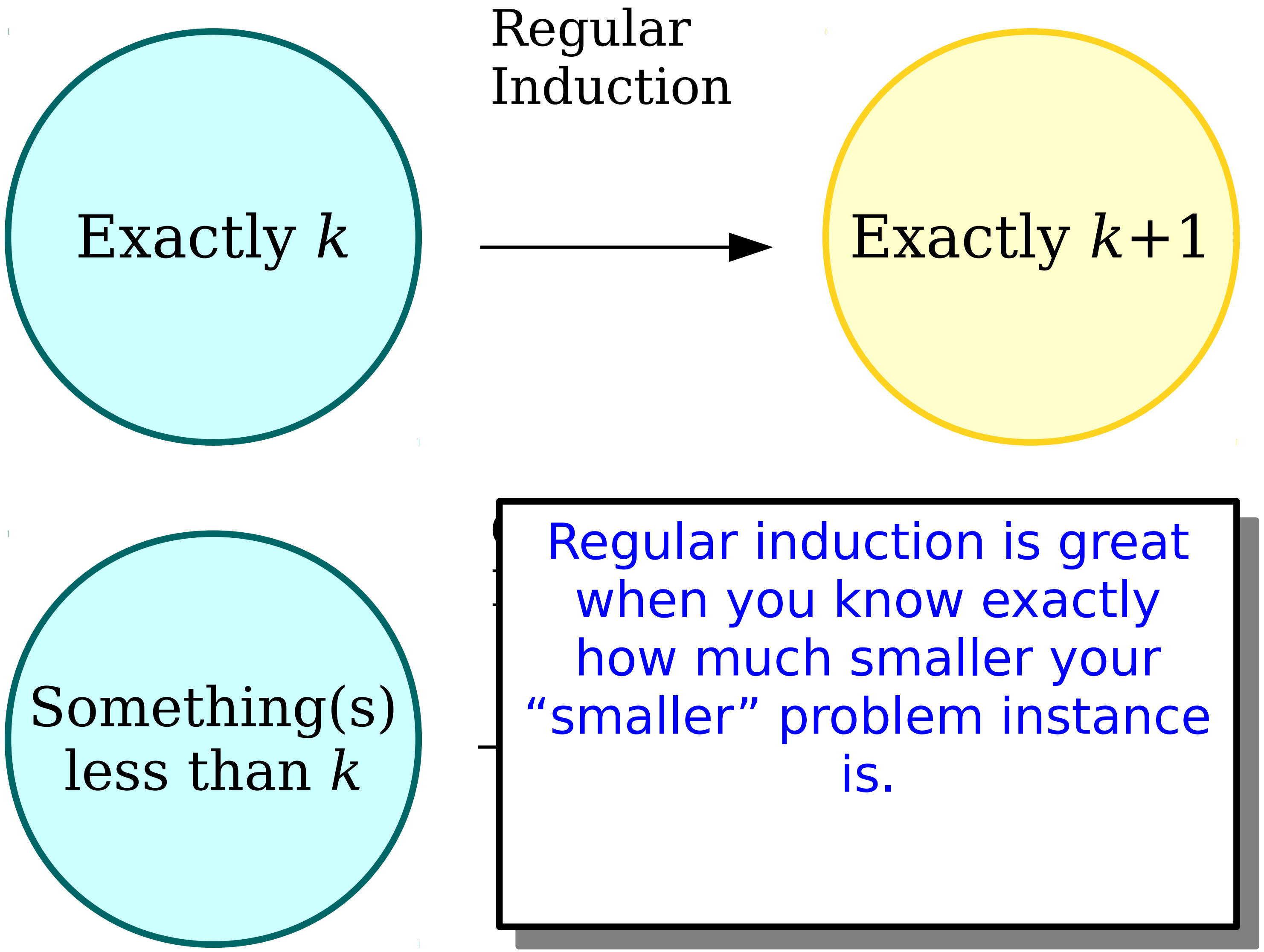


- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

Induction vs. Complete Induction



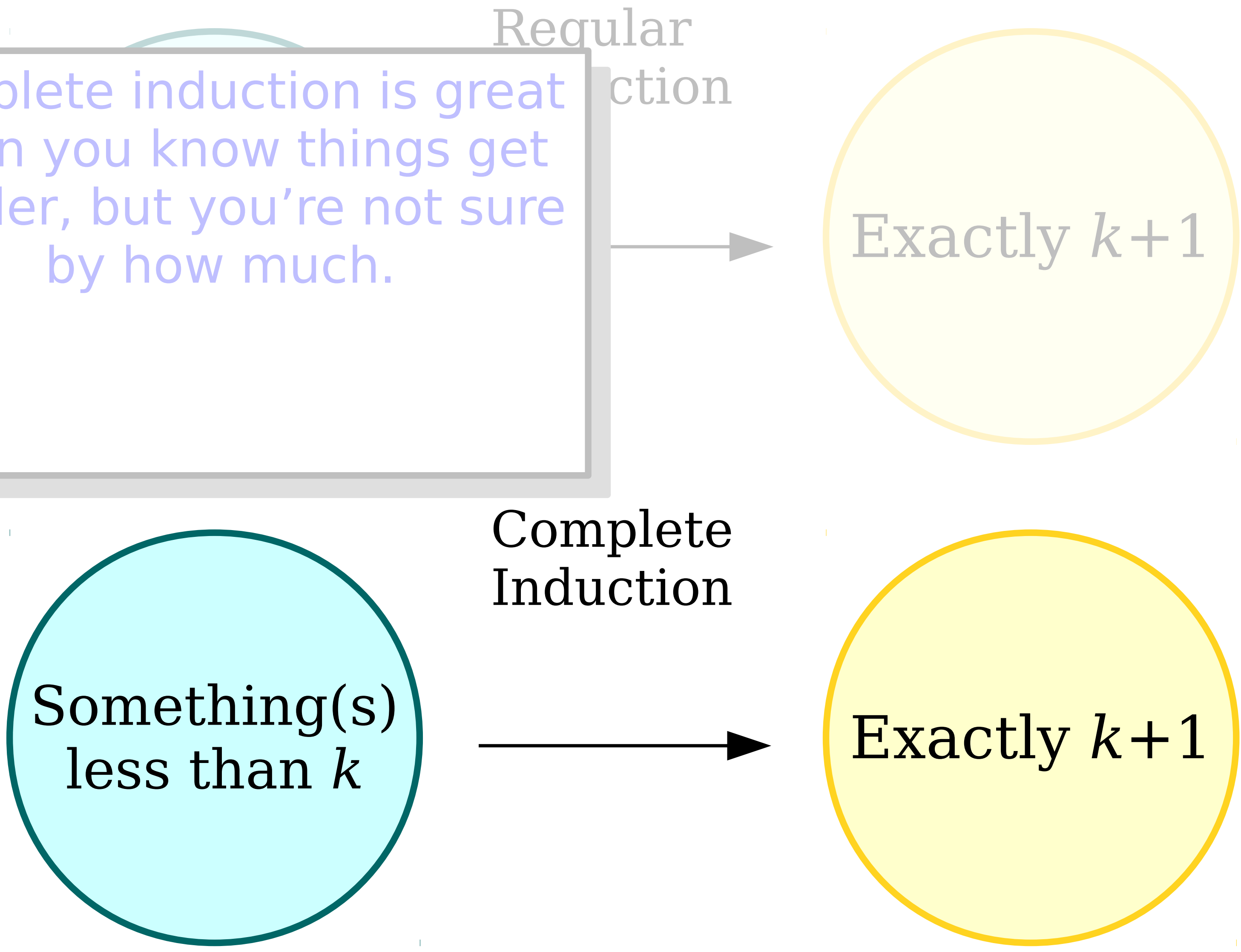
Induction vs. Complete Induction



Regular induction is great when you know exactly how much smaller your "smaller" problem instance is.

Induction vs. Complete Induction

Complete induction is great when you know things get smaller, but you're not sure by how much.



How Not to Induct

All Horses are the Same Color

$P(n)$ = “All groups of n horses always have the same color”

All Horses are the Same Color

$P(0)$ = "All groups of 0 horses always have the same color"

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = “All groups of k horses always have the same color”

(for the sake of illustration, pretend arbitrarily picked $k = 5$)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

(for the sake of illustration, pretend arbitrarily picked $k = 5$)
(so $k + 1$ group is 6)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

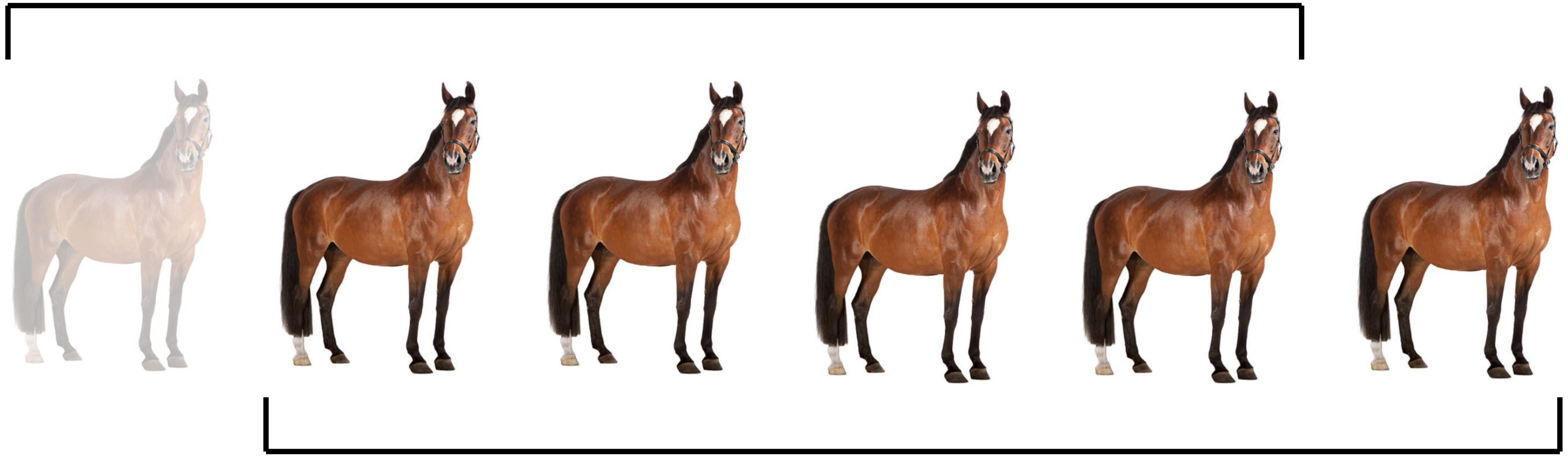
By $P(k)$, these k horses have the same color

(for the sake of illustration, pretend arbitrarily picked $k = 5$)
(so $k + 1$ group is 6)

All Horses are the Same Color

Prove $P(k{+}1)$ = “All groups of $k{+}1$ horses always have the same color”

By $P(k)$, these k horses have the same color



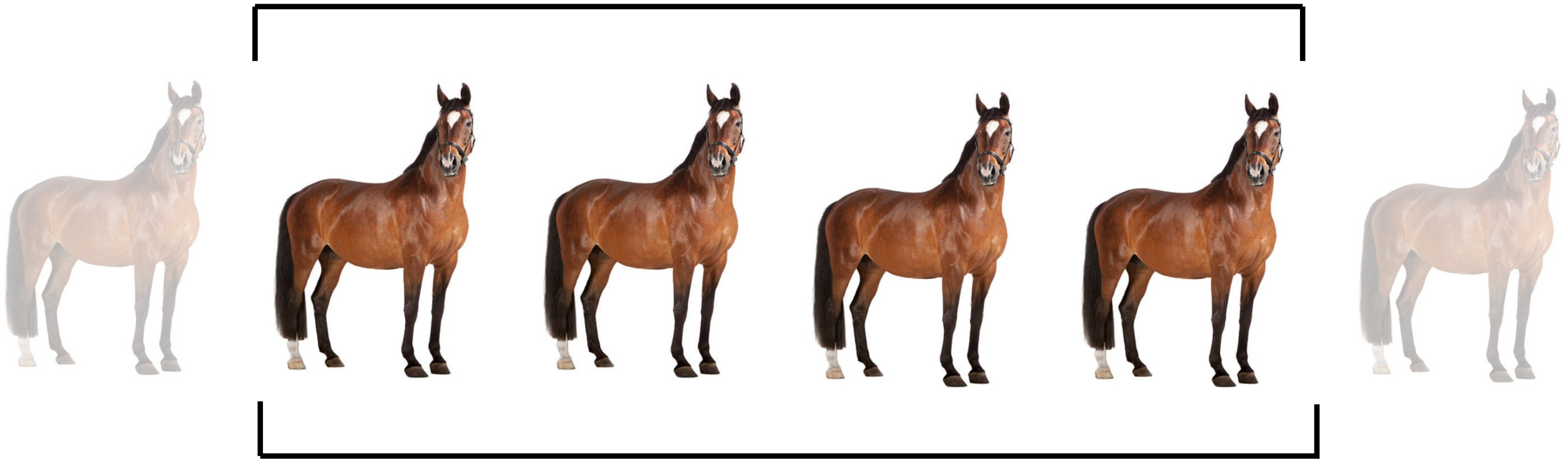
By $P(k)$, these k horses have the same color

(for the sake of illustration, pretend arbitrarily picked $k = 5$)
(so $k + 1$ group is 6)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets



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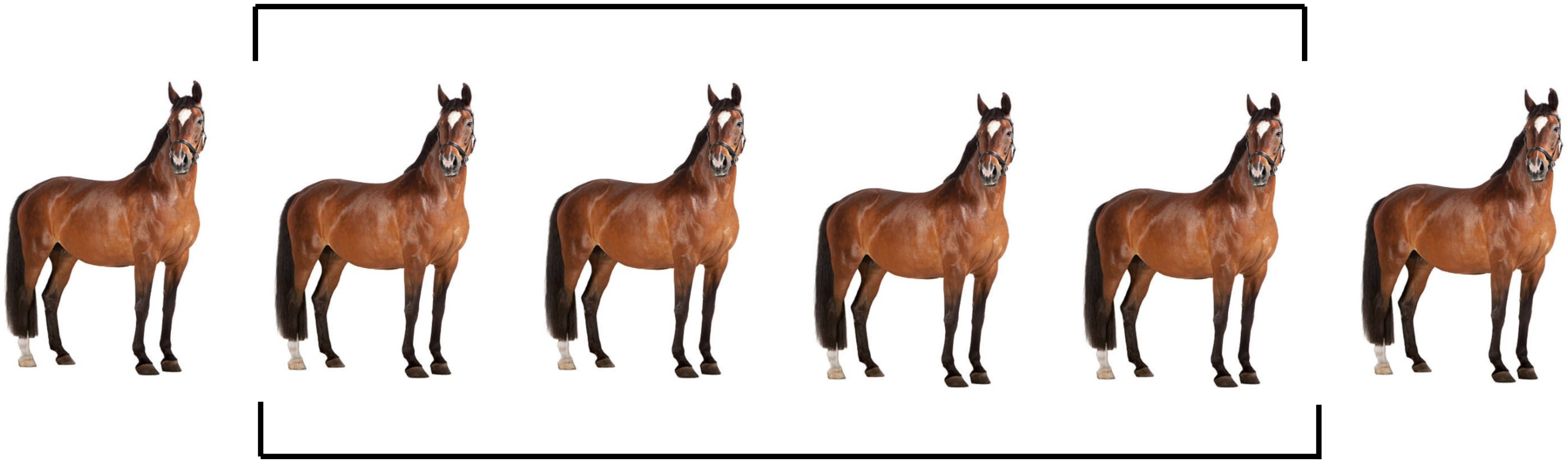

And we said that both horses on the ends are the same color as these overlapping horses

(for the sake of illustration, pretend arbitrarily picked $k = 5$)
(so $k + 1$ group is 6)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

So all $k+1$ horses have the same color!

(for the sake of illustration, pretend arbitrarily picked $k = 5$)
(so $k + 1$ group is 6)

⚠ ***Incorrect!*** ⚠ ***Proof:*** Let $P(n)$ be the statement "all groups of n horses are the same color." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

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1

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

2

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

3

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

4

Quick check: Which paragraph has the error?
Go to PollEv.com/cs103spr25

What’s going on here?

⚠ ***Incorrect!*** ⚠ ***Proof:*** Let $P(n)$ be the statement "all groups of n horses are the same color." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

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For the inductiv … al number k
that $P(k)$ is true … me color. Now
consider a grou … d look only at
the first k horse … es are the
same color. Ne … the last k
horses. Again w … se horses are
the same color.

The logic in our inductive step does not allow us to get from *P*(1) to $P(2)$.
Specifically, there are no non-excluded horses that were in both sets.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

All Horses are the Same Color

Assume $P(k)$ = “All groups of k horses always have the same color”

(now let’s pretend arbitrarily picked $k = 1$)

All Horses are the Same Color

Prove $P(k{+}1)$ = "All groups of $k{+}1$ horses always have the same color"

(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 2)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

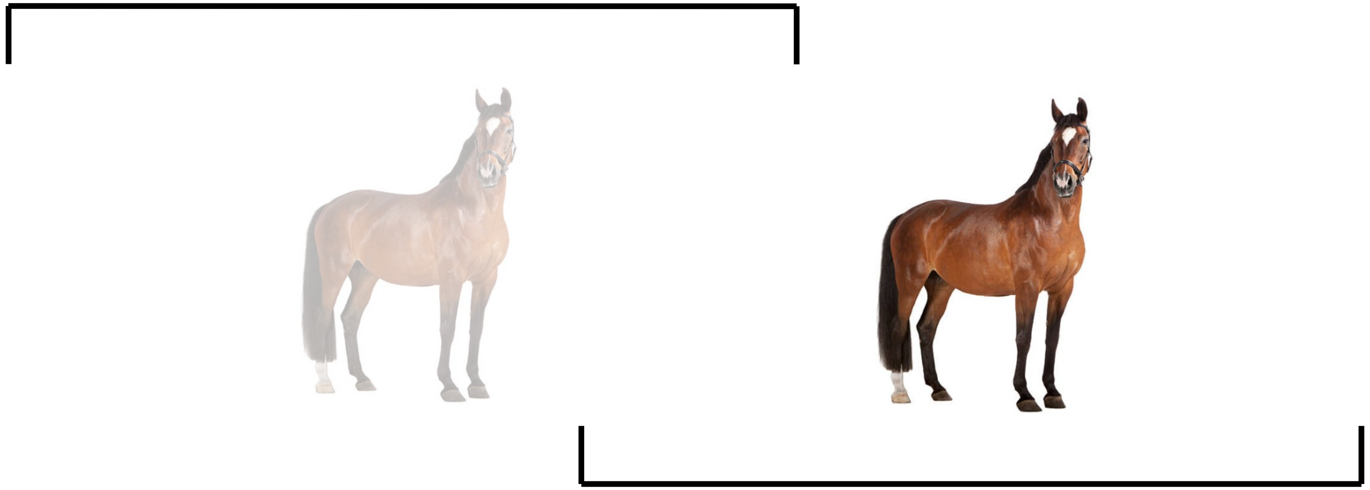
By $P(k)$, these k horses have the same color

(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 1)

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color



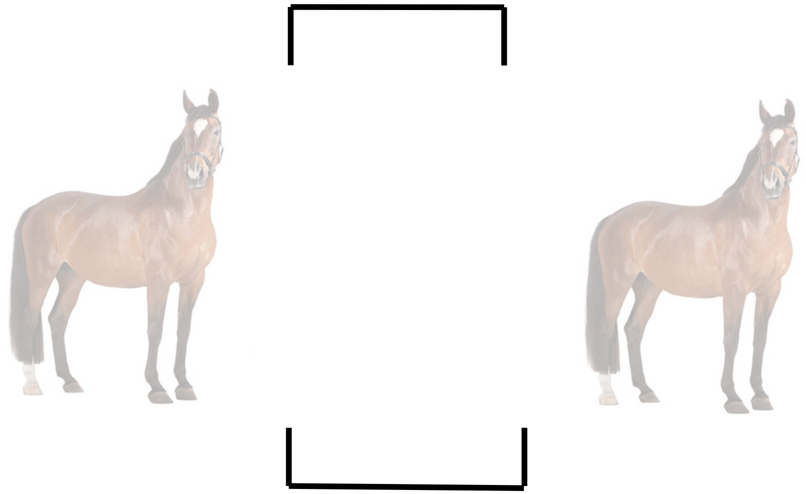
By $P(k)$, these k horses have the same color

(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 2)

All Horses are the Same Color

Prove $P(k{+}1)$ = “All groups of $k{+}1$ horses always have the same color”

The horses in the middle were in both sets



(for the sake of illustration, pretend arbitrarily picked $k = 1$)
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All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

The horses in the middle were in both sets

And we said that both horses on the ends are the same color as these overlapping horses

(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 2)

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

So all $k+1$ horses have the same color!

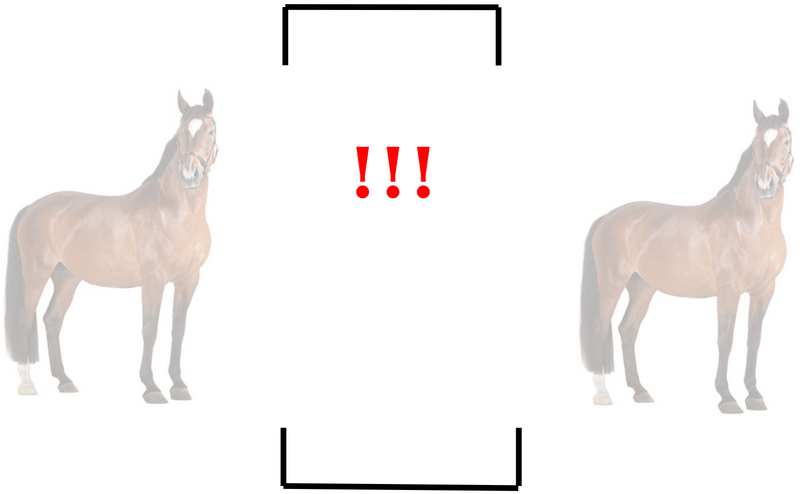
(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 2)

All Horses are the Same Color

Procedure the proof describes does not work for $k < 2$.

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

The horses in the middle were in both sets



(for the sake of illustration, pretend arbitrarily picked $k = 1$)
(so $k + 1$ group is 2)

An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.